# Frequency-weighted damping via nonsmooth optimization and fast computation of QEPs with low-rank updates

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# Introduction and motivation

We consider damped linear vibration system

$$M\ddot{q}(t) + C(v)\dot{q}(t) + Kq(t) = f(t),$$
$$q(0) = q_0, \quad \text{and} \quad \dot{q}(0) = \dot{q}_0.$$

Where $M, C(v), K \in \mathbb{R}^{n\times n}$ system matrices, $v \in \mathbb{R}^s$ parameter vector.

- $M, K$ are positive definite Herimitian matrices
- $C(v) = C_{int} + C_{ext}(v)$, $C_{int} > 0$ internal damping, $C_{ext}(v) \geq 0$ external damping.
- $C_{ext}(v) = \sum_{i=1}^{s} v_i g_i g_i^T$
- $C_{int} = \alpha_c C_{crit}$, where $C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}$.

## Linearization

- Let $\Phi$ simultaneously diagonalize pair $M$ and $K$

$$\Phi^T K \Phi = \Omega^2 = \operatorname{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I.$$

Note that internal damping is s.t. $\Phi^T C_{int} \Phi = \alpha\Omega$.
With $q = \Phi q_\Phi$ and $y_1 = \Omega q_\Phi$, $y_2 = \dot{q}_\Phi$ we have

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T C(v) \Phi \end{bmatrix}}_{A(v)} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

We obtain first order differential equation:

$\dot{y} = Ay$, with solution $y = e^{At} y_0$, where $y_0$ is initial data.

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We consider parameter dependant QEP:

$$\left(\lambda^2(v)M + \lambda(v)C(v) + K\right) x(v) = 0.$$

With $w_1(v) = \Omega\Phi^{-1}x(v)$ and $w_2(v) = \lambda(v)\Omega^{-1}w_1(v)$ we have

$$\lambda(v)\begin{bmatrix} w_1(v) \\ w_2(v) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T C(v)\Phi \end{bmatrix}}_{A(v)} \begin{bmatrix} w_1(v) \\ w_2(v) \end{bmatrix}.$$

**Very important question arises in considering such systems:**

For the given mass (M) and stiffness (K) determine the best (optimal) damping which will insure optimal evanescence.

$(M, K)$, $A$ not stable

$(M, K, C)$, $A$ stable

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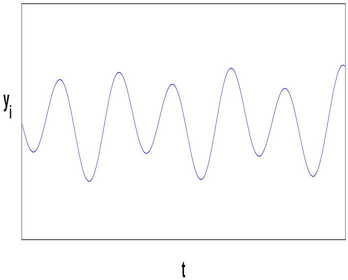


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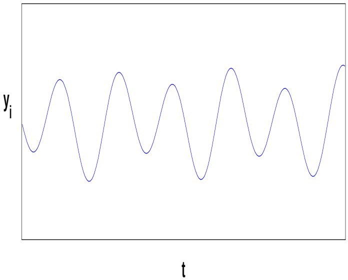


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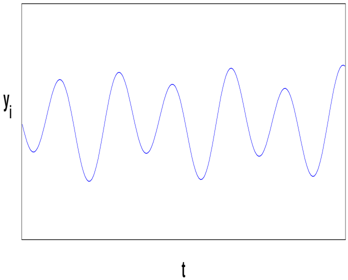


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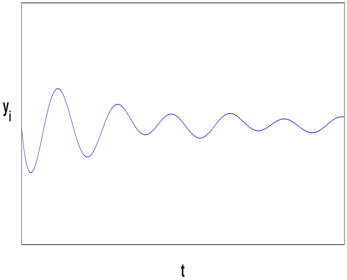

**Very important question arises in considering such systems:**

How to re-design a given damped mechanical system, such that a new system does not have eigenvalues in some ”dangerous” part of a complex plane, typically called the resonance band.

## Optimization criteria

- Minimization of spectral abscissa

$$\alpha_{MCK}(v) \quad \to \quad \min_{v},$$

where $\alpha_{MCK}(v) = \max_k \operatorname{Re} \lambda_k(v)$ and $\lambda_k(v)$ are the eigenvalues of

$$\left(\lambda^2(v)M + \lambda(v)C(v) + K\right) x(v) = 0.$$

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# Optimization criteria-idea

The undesirable frequency band: $[\omega - b, \omega + b]$, for some given $b > 0$, where $\omega \in \mathbb{R}$ is an undesirable frequency.

$$\min_{v\in\mathbb{R}^s} \quad \max\{\operatorname{Re}\lambda(v) : \lambda(v) \in \Lambda(v) \text{ and } \operatorname{Im}\lambda(v) \in [\omega - b, \omega + b]\}$$

$$\text{s.t.} \quad \alpha_{\mathrm{MCK}}(v) \leq \mathtt{tol}_{\mathrm{sa}} \text{ for some } \mathtt{tol}_{\mathrm{sa}} < 0,$$

$$v_j \geq 0 \text{ for } j = 1, \ldots, s$$

# Using ellipse instead of band

Let $E = (a, b, c)$ denote the axis-aligned ellipse

$$\frac{(x - \operatorname{Re} c)^2}{a^2} + \frac{(y - \operatorname{Im} c)^2}{b^2} = 1,$$

where $a, b > 0$ respectively denote the semi-major and -minor axes and $c \in \mathbb{C}$ is the center of the ellipse. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, consider the following algebraic distance $d : \mathbb{C} \mapsto [0, \infty)$ of a point $z \in \mathbb{C}$ to this ellipse, i.e.,

$$d(z; E) := \frac{(\operatorname{Re}(z - c))^2}{a^2} + \frac{(\operatorname{Im}(z - c))^2}{b^2}.$$

# Using ellipse instead of band

Measure of the distance of the spectrum to the undesirable frequency $[\omega - b, \omega + b]$, we define

$$d_{\Lambda,E}(v) \coloneqq \min\{d(\lambda(v); E) : \lambda(v) \in \Lambda(v)\},$$

where $E = (a, b, \mathbf{i}\omega)$. Similarly,

$$d_{\Lambda,\mathcal{E}}(v) \coloneqq \min\{d_{\Lambda,E_j}(v) : E_j \in \mathcal{E}\},$$

where $E_j \coloneqq (a_j, b_j, \mathbf{i}\omega_j)$ is defining the $j$th ellipse for the $j$th undesirable frequency band $[\omega_j - b_j, \omega_j + b_j]$ with relative importance $a_j > 0$ and $\mathcal{E} \coloneqq \{E_1, \ldots, E_k\}$ is the set of $k$ corresponding ellipses.

# New optimization criteria

- Frequency isolation while minimizing spectral abscissa (FI1)

**FI1:**
$$\begin{aligned} \min_{v\in\mathbb{R}^s} \quad & \alpha_{\mathrm{MCK}}(v) \\ \text{s.t.} \quad & d_{\Lambda,\mathcal{E}}(v) \geq 1, \\ & \alpha_{\mathrm{MCK}}(v) \leq \mathtt{tol}_{\mathrm{sa}} \text{ for some } \mathtt{tol}_{\mathrm{sa}} < 0, \\ & v_j \geq 0 \text{ for } j = 1, \ldots, s. \end{aligned}$$

## New optimization criteria

- Frequency isolation while maximizing the major axis of the ellipses (FI2)

**FI2:**
$$\max_{v\in\mathbb{R}^s} \sum_{j=1}^{k} \phi_j a_{\Lambda,E_j}(v)$$
s.t. $\alpha_{\mathrm{MCK}}(v) \le \mathtt{tol}_{\mathrm{sa}}$ for some $\mathtt{tol}_{\mathrm{sa}} < 0$,
$v_j \ge 0$ for $j = 1,\dots,s$.

where $a_{\Lambda,E}(v) := \min\{a(\lambda(v);E) : \lambda(v) \in \Lambda(v)\}$,

$$a(z;E) := \begin{cases} \frac{b|\mathrm{Re}\, z|}{\sqrt{b^2-(\mathrm{Im}\, z-\omega)^2}}, & \text{if } \mathrm{Im}\, z \in (\omega-b,\omega+b), \\ \infty & \text{otherwise.} \end{cases}$$

# GRANSO: GRadient-based Algorithm for Non-Smooth Optimization

Requires gradients:

$$\begin{aligned}
\left.\frac{\partial \lambda(v)}{\partial v_j}\right|_{v=\hat{v}} &= -\frac{\hat{x}^* \left(\lambda(\hat{v}) g_j g_j^T\right) \hat{x}}{\hat{x}^*(2\lambda(\hat{v})M + C(\hat{v}))\hat{x}}, \\
\left.\frac{\partial \alpha_{\mathrm{MCK}}(v)}{\partial v_j}\right|_{v=\hat{v}} &= -\mathrm{Re} \left.\frac{\partial \lambda(v)}{\partial v_j}\right|_{v=\hat{v}}, \\
d'(z(t);E) &= 2\left(\frac{\mathrm{Re}(z(t)-c)\cdot \mathrm{Re}z'(t)}{a^2} + \frac{\mathrm{Im}(z(t)-c)\cdot \mathrm{Im}z'(t)}{b^2}\right), \\
a'(z(t);E) &= \frac{b\,\mathrm{sgn}(\mathrm{Re}\, z(t))\cdot \mathrm{Re}\, z'(t)}{(b^2-(\mathrm{Im}\, z(t)-\omega)^2)^{1/2}} + \frac{b|\mathrm{Re}\, z(t)|(\mathrm{Im}\, z(t)-\omega)\cdot \mathrm{Im}\, z'(t)}{(b^2-(\mathrm{Im}\, z(t)-\omega)^2)^{3/2}}.
\end{aligned}$$

Apply $s$ times efficient algorithm for computing eigenvalues of (CSymDPR1) matrix $A = D + \rho uu^T$ .

The eigenvalues of $A$ are the zeros of the secular function:

$$f(\lambda) = 1 + \rho \sum_{i=1}^{2n} \frac{u_i^2}{d_i - \lambda} = 1 + \rho u^T (D - \lambda I)^{-1} u,$$

and the corresponding eigenvectors are given by

$$w_i = \frac{y_i}{\|y_i\|_2}, \text{ where } y_i = (D - \lambda_i I)^{-1} u, \quad i = 1, \ldots, 2n.$$

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initial vector $y_0 \neq 0$

Rayleigh quotient iteration
$\lambda = 0$

until convergence

1. $y^1 := (A - \lambda I)^{-1} y^0$
2. $\mu = \frac{{y^1}^T y^0}{{y^0}^T y^0}$
3. $\lambda = \lambda + \frac{1}{\mu}$
4. $y^0 = y^1$

Modified Rayleigh quotient iteration
$\lambda = 0$

until convergence

1. $\mu = \eta \frac{{y^0}^T (A - \lambda I) y^0}{{y^0}^T y^0}$
2. $\lambda = \lambda + \mu$
3. $y^0 := (D - \lambda I)^{-1} u$

compute eigenvectors

N. Jakovčević Stor, I. Slapničar, and Z. Tomljanović. *Fast computation of optimal damping parameters for linear vibrational systems.arXiv e-prints*, 2020

$$C_{ext}(v) = \sum_{i=1}^{s} v_i g_i g_i^T = G\text{diag}(v)G^T$$

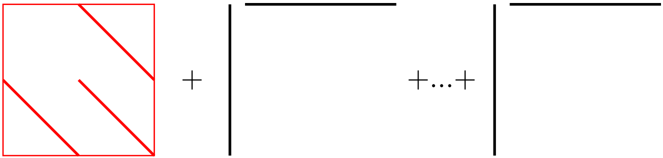
How to get from $Ay = \lambda y$, where

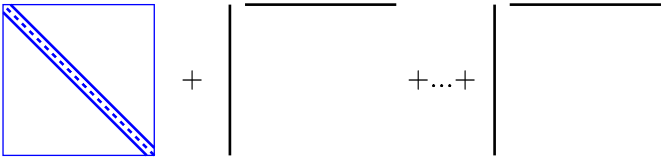
$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\alpha\Omega - \Phi^T G\text{diag}(v)G^T\Phi \end{bmatrix},$$

to multiple CSymDPR1 eigenvalue problem?

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\alpha\Omega \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\Phi^T G \operatorname{diag}(v) G^T \Phi \end{bmatrix}$$

$$P^T \cdot \quad \backslash \quad A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\alpha\Omega \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\Phi^T G \mathrm{diag}(v) G^T \Phi \end{bmatrix} \quad / \quad \cdot P$$



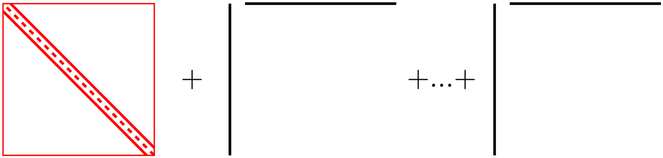


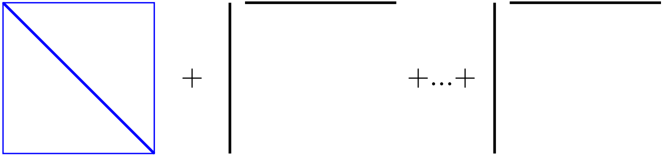
$$P^T AP = \left( \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{bmatrix} - \widehat{G} \begin{bmatrix} v_1 & & \\ & \ddots & \\ & & v_s \end{bmatrix} \widehat{G} \right), \quad \text{where}$$

$$D_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -\alpha\omega_i \end{bmatrix} \quad \text{and} \quad \widehat{G} = P^T \begin{bmatrix} 0 \\ \Phi^T G \end{bmatrix}.$$

$$\Psi^{-1} \cdot \quad \backslash \quad P^T A P \; = \left( \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{bmatrix} - \widehat{G} \begin{bmatrix} v_1 & & \\ & \ddots & \\ & & v_s \end{bmatrix} \widehat{G} \right), \quad / \quad \cdot \, \Psi$$

where $\Psi = \begin{bmatrix} \Psi_1 & & \\ & \ddots & \\ & & \Psi_n \end{bmatrix}$

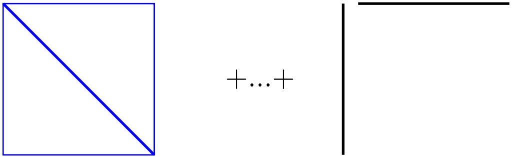


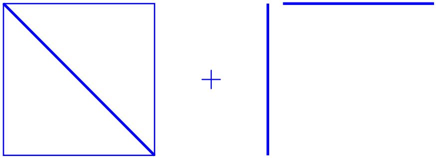


$$\widetilde{A} = D - U \begin{bmatrix} v_1 & & \\ & \ddots & \\ & & v_s \end{bmatrix} Z^T = D - \sum_{j=1}^{s} v_j u_j z_j^T,$$

$$D = \Psi^{-1} \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{bmatrix} \Psi,$$

$$U = \Psi^{-1} \widehat{G}, \quad \text{and} \quad Z = \Psi^T \widehat{G},$$





**Algorithm 1** Frequency-weighted damping optimization algorithm

**Input:** $M$ and $K$, $\alpha \geq 0$ for $C_{\text{int}}$ and $G$, set of $k$ ellipses $\mathcal{E}$, weights $[\phi_1, \ldots, \phi_k]$ with each $\phi_j \in (0, 1]$ for ellipse $E_j \in \mathcal{E}$, $\texttt{tol}_{\text{sa}} < 0$, initial viscosity $v_{\text{init}} \in \mathbb{R}^s_+$, and $\texttt{approach} \in \{1, 2\}$.

**Output:** Computed for optimized viscosities $v_{\text{opt}} \in \mathbb{R}^s_+$ for either FI1 or FI2

1: $[\Phi, \Omega]$ matrices from linearization
2: $[\Psi, D, U, Z]$ matrices that construct low-rank structure
3: **if** $\texttt{approach} = 1$ **then**
4: $v_{\text{opt}} \leftarrow$ solution returned by GRANSO for FI1 initialized at $v_{\text{init}}$
5: **else**
6: $v_{\text{opt}} \leftarrow$ solution returned by GRANSO for FI2 initialized at $v_{\text{init}}$
7: **end if**

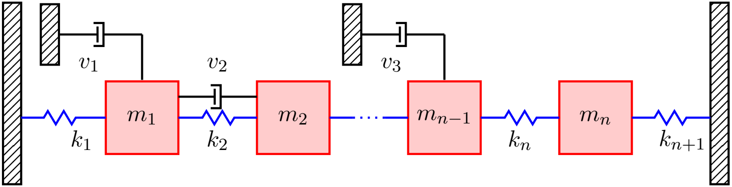



Figure: $n$-mass oscillator

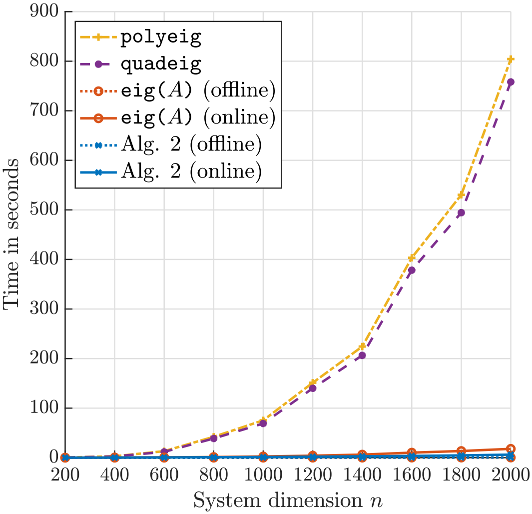
$$
\begin{aligned}
n &= 200 \cdot i, \quad i = 1, \ldots, 10, \\
M &= \operatorname{diag}(m_1, m_2, \ldots, m_n), \\
m_i &= 10 + \frac{990}{n-1} \cdot (i-1), i = 1, \ldots, n
\end{aligned}
$$

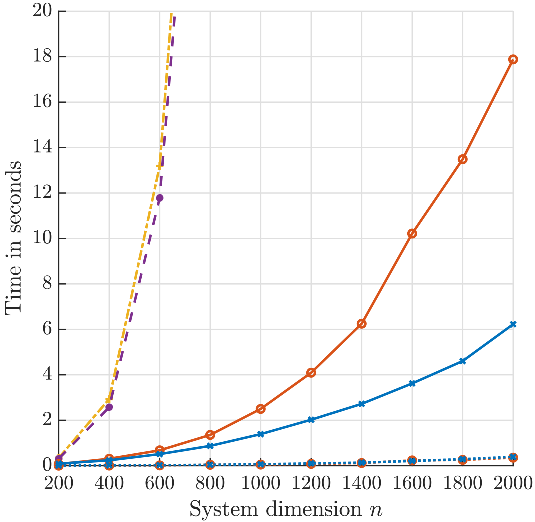
# Example

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & \ddots & \ddots & & \\ & \ddots & \ddots & & -k_n \\ & & & -k_n & k_n+k_{n+1} \end{bmatrix}, \quad k_i = 5, i = 1, \dots, n+1$$

$$C_{\text{ext}} = v_1 e_k e_k^T + v_2(e_j - e_{j+1})(e_j - e_{j+1})^T + v_3 e_l e_l^T, \alpha = 0.004$$
$$v = [v_1, v_2, v_3]^T, v_1, v_2, v_3 \in [0.1, 1.1]$$

$$(k, j, l) = \left(\frac{n}{10}, \frac{3n}{10}, \frac{5n}{10}\right)$$


polyeig
quadeig
eig($A$) (offline)
eig($A$) (online)
Alg. 2 (offline)
Alg. 2 (online)
Time in seconds
0
100
200
300
400
500
600
700
800
900
200
400
600
800
1000
1200
1400
1600
1800
2000
System dimension $n$


Time in seconds
0
2
4
6
8
10
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14
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18
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200
400
600
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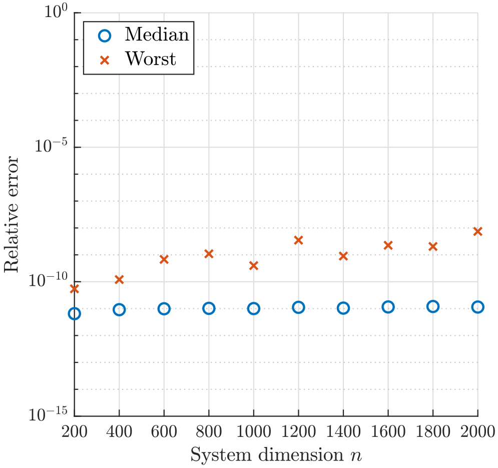
1000
1200
1400
1600
1800
2000
System dimension $n$



Median
Worst
Relative error
$10^{0}$
$10^{-5}$
$10^{-10}$
$10^{-15}$
200 400 600 800 1000 1200 1400 1600 1800 2000
System dimension $n$

# Example

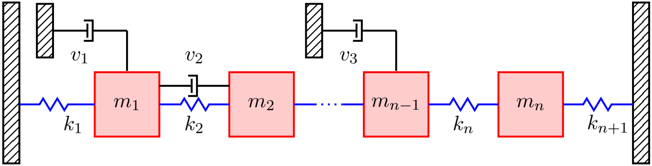
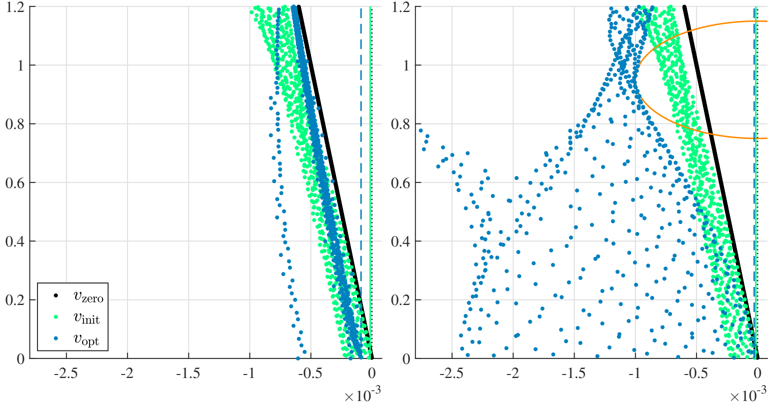



Figure: $n$-mass oscillator

$$
\begin{aligned}
n &= 1000, \\
m_i &= m_{n+1-i} = \frac{2n-i}{200} \quad \text{for} \quad i = 1, \ldots, \frac{n}{2}, \\
k_i &= 5 \quad \text{for} \quad i = 1, \ldots, n+1, \\
(j, k, l) &= (100, 400, 900), \\
v_{\text{init}} &= \texttt{ones(3,1)}.
\end{aligned}
$$

$$E = (0.001, 0.2, 0.95\mathbf{i})$$



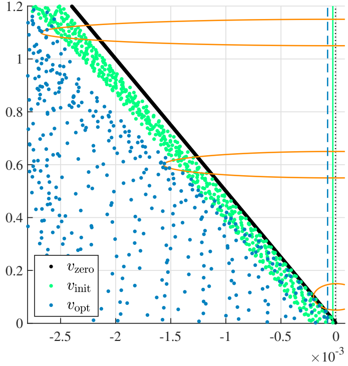

(a) Spectral abscissa minimization only

$$v_{\text{opt}} = \begin{bmatrix} 314.7 \\ 94.1 \\ 124.5 \end{bmatrix}$$

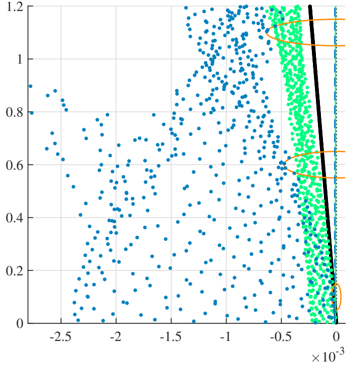
(b) Approach 1 (Fixed ellipses)

$$v_{\text{opt}} = \begin{bmatrix} 8.032 \\ 8.184 \\ 9.970 \end{bmatrix}$$

$$E_1 = (\sim, 0.05, 0.1\mathbf{i}), \phi_1 = 1, \quad E_2 = (\sim, 0.05, 0.6\mathbf{i}), \phi_2 = 0.2, \quad E_3 = (\sim, 0.05, 1.1\mathbf{i}), \phi_3 = 0.1$$




(c) $\alpha = 0.004$, $v_{opt} = \begin{bmatrix} 8.137 \\ 7.147 \\ 1.789 \end{bmatrix}$




(d) $\alpha = 0.0004$, $v_{opt} = \begin{bmatrix} 8.294 \\ 7.767 \\ 1.673 \end{bmatrix}$

# Thank you for your attention!