Some contributions to output controllability

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General Problem

We consider the finite dimensional system

$$\dot{x} = Ax + Bu, \qquad x(0) = x^0, \tag{1a}$$

with output

$$y = Cx + Du, \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$.

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Problem (Output and Long-time output controllability)

Given $x^0 \in \mathbb{R}^n$ *and* $y^1 \in \mathbb{R}^q$ *and* $T > 0$*,*

Find $u \in C^0([0, T], \mathbb{R}^m)$ *such that the solution of* (1) *satisfies:* $y(T) = y^1,$ **Output controllability**	*Find* $u \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ *such that the solution of* (1) *satisfies:* $y(t) = y^1, \quad \forall t \geqslant T.$ **Long-time output controllability**

Output controllability $\Leftarrow$ Long-time output controllability

State of art

The notion of output controllability first appears in

- J. Bertram and P. Sarachik. "On optimal computer control". *IFAC Proceedings Volumes* 1.1 (1960)

Conditions in terms of Gramian and rank are given in

- E. Kreindler and P. E. Sarachik. "On the concepts of controllability and observability of linear systems". *IEEE Trans. Automatic Control* AC-9 (1964)

Some of these results have been reported in

- C.-T. Chen. *Introduction to linear system theory*. 1970
- H. D'Angelo. *Linear time-varying systems: Analysis and synthesis*. 1970

The notion of long-time output controllability has to be related to the ones of *functional output controllability*,

- W. Wonham. *Linear multivariable control. A geometric approach*. 1985
- G. Basile and G. Marro. *Controlled and conditioned invariants in linear system theory*. 1992

and *dead-beat control*.

- T. R. Crossley and B. Porter. "Dead-beat control of sampled-data systems with bounded input". *Internat. J. Control* 19 (1974)

Elimination of D I

Remark (A necessary condition)

$$\text{Output controllability} \quad \Rightarrow \quad \operatorname{rk}\begin{pmatrix} C & D \end{pmatrix} = q.$$

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Lemma

The system (1) *is output controllable if and only if the system*

$$\dot{X} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ I_m \end{pmatrix} v, \tag{2a}$$

$$y = \begin{pmatrix} C & D \end{pmatrix} X \tag{2b}$$

is output controllable.

Remark (A sufficient condition)

$$\text{State controllability and } \mathrm{rk}\begin{pmatrix} C & D \end{pmatrix} = q \quad \Rightarrow \quad \text{output controllability.}$$

Elimination of D II

Proof of the Lemma.

- Of course, (2) is output controllable implies that (1) is output controllable.
- Reciprocally, if (1) is output controllable, then for every $x^0 \in \mathbb{R}^n$ and every $y^1 \in \mathbb{R}^q$, there exist $x^1 \in \{e^{TA}x^0\} + \operatorname{Im}\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$ and $u^1 \in \mathbb{R}^m$ such that

$$y^1 = Cx^1 + Du^1.$$

  Furthermore, for every $u^0 \in \mathbb{R}^m$, there exist $u \in C^1([0,T], \mathbb{R}^m)$ such that $u(0) = u^0$, $u(T) = u^1$ and $x(T; u, x^0) = x^1$.
  We conclude by taking $v = \dot{u}$ in (2). □

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We conclude by taking $v = \dot{u}$ in (2). □

As consequence, we only consider (1) with $D = 0$.

In addition, instead of considering a control $u \in C^0([0, T], \mathbb{R}^m)$, we will consider $u \in L^2([0, T], \mathbb{R}^m)$.

Results of Kreindler and Sarachik[2]

Theorem

The following properties are equivalent.

0. *The system* (1) *is output controllable.*
1. $\operatorname{rk}\left(C\begin{pmatrix}B & AB & \dots & A^{n-1}B\end{pmatrix}\right) = q$.
2. $\mathcal{K}_T := C\displaystyle\int_0^T e^{(T-t)A}BB^\top e^{(T-t)A^\top}\,\mathrm{d}t\,C^\top \in \mathbb{R}^{q\times q}$ *is positive definite for some* $T > 0$.

[1] R. E. Kalman. "Contributions to the theory of optimal control". *Bol. Soc. Mat. Mexicana (2)* 5 (1960)

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With $q = n$ and $C = I_n$, we have:

1. $\operatorname{rk}\begin{pmatrix}B & AB & \dots & A^{n-1}B\end{pmatrix} = n$.
2. $\int_0^T e^{(T-t)A}BB^\top e^{(T-t)A^\top}\,\mathrm{d}t \in \mathbb{R}^{n\times n}$ is positive definite for some $T>0$.

These are the necessary and sufficient conditions for (state) controllability of Kalman[1].

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The Hautus test is missing.

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[2] E. Kreindler and P. E. Sarachik. "On the concepts of controllability and observability of linear systems". *IEEE Trans. Automatic Control* AC-9 (1964)

Hautus test for (state) controllability

Theorem

The following properties are equivalent.

0. *The system* (1a) *is (state) controllable.*
3. $\ker(B^\top) \cap \ker(A^\top - \lambda I_n) = \{0\}, \quad \forall \lambda \in \mathbb{C}.$
4. $\mathrm{rk}\begin{pmatrix} A - \lambda I_n & B \end{pmatrix} = n, \quad \forall \lambda \in \mathbb{C}.$

Remark

- This result is also known as Popov-Belevitch-Hautus condition[3,4,5].
- Doing the test for $\lambda \in \mathbb{C}$ or for $\lambda \in \sigma(A)$ is the same.

[3] V. Belevitch. *Classical network theory*. 1968

[4] M. L. J. Hautus. "Controllability and observability conditions of linear autonomous systems". *Nederl. Akad. Wetensch. Proc. Ser. A 72 = Indag. Math.* 31 (1969)

[5] V.-M. Popov. *Hyperstability of control systems*. Translated from the Romanian by Radu Georgescu, Die Grundlehren der mathematischen Wissenschaften, Band 204. 1973

Hautus test for output controllability I

Theorem

The following properties are equivalent.

0. *The system* (1) *is output controllable.*

3. $\operatorname{rk} C = q \quad \text{and} \quad \operatorname{Im} C^\top \cap \left(\bigoplus_{\lambda \in \sigma(A)} E_\lambda \right) = \{0\}$,

   where $E_\lambda = \ker(A^\top - \lambda I_n)^{n_\lambda} \cap \left(\bigcap_{k=0}^{n_\lambda - 1} \ker B^\top (A^\top - \lambda I_n)^k \right)$,

   with n_λ, *the algebraic multiplicity of* λ *in the minimal polynomial of* A.

4. $\operatorname{rk} C = q \quad \text{and} \quad \operatorname{rk} \begin{pmatrix} M_{\lambda_1} & & 0 & C^\perp \\ & \ddots & & \vdots \\ 0 & & M_{\lambda_p} & C^\perp \end{pmatrix} = np$,

   where $p = \#\sigma(A)$, $\{\lambda_1, \lambda_2, \cdots, \lambda_p\} = \sigma(A)$, *and*

   $M_\lambda = \begin{pmatrix} (A - \lambda I_n)^{n_\lambda} & (A - \lambda I_n)^{n_\lambda - 1} B & \dots & (A - \lambda I_n) B & B \end{pmatrix} \in \mathbb{R}^{n \times (n + n_\lambda m)}$.

   $C^\perp \in \mathbb{R}^{n \times \dim \ker C}$ *is a matrix of maximal rank such that* $C C^\perp = 0$.

Hautus test for output controllability II

When $C = I_n$

Let us consider the case $n = q$ and $C = I_n$. We have

3. $\bigoplus_{\lambda\in\sigma(A)} E_\lambda = \{0\} \quad \Rightarrow \quad \forall \lambda \in \sigma(A),\ E_\lambda = \{0\}$

$$\Rightarrow \quad \forall \lambda \in \sigma(A),\ \ker(A^\top - \lambda I_n)^{n_\lambda} \cap \left(\bigcap_{k=0}^{n_\lambda - 1} \ker B^\top (A^\top - \lambda I_n)^k\right) = \{0\}$$

$$\Rightarrow \quad \forall \lambda \in \sigma(A),\ \ker(A^\top - \lambda I_n) \cap \ker B^\top = \{0\}.$$

4. $\mathrm{rk} \begin{pmatrix} M_{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & M_{\lambda_p} \end{pmatrix} = np \quad \Rightarrow \quad \forall \lambda \in \sigma(A),\ \mathrm{rk}\, M_\lambda = n$

$$\Rightarrow \quad \forall \lambda \in \sigma(A),\ \mathrm{rk}\left((A - \lambda I_n)^{n_\lambda} \quad (A - \lambda I_n)^{n_\lambda - 1} B \quad \dots \quad (A - \lambda I_n) B \quad B\right) = n$$

$$\Rightarrow \quad \forall \lambda \in \sigma(A),\ \ker(A^\top - \lambda I_n)^{n_\lambda} \cap \left(\bigcap_{k=0}^{n_\lambda - 1} \ker B^\top (A^\top - \lambda I_n)^k\right) = \{0\}$$

$$\Rightarrow \quad \forall \lambda \in \sigma(A),\ \ker(A^\top - \lambda I_n) \cap \ker B^\top = \{0\}$$

$$\Rightarrow \quad \forall \lambda \in \sigma(A),\ \mathrm{rk}\left(A - \lambda I_n \quad B\right) = n.$$

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$$\Rightarrow \forall\lambda\in\sigma(A),\ \ker(A^\top - \lambda I_n) \cap \ker B^\top = \{0\}.$$

4. $\operatorname{rk}\begin{pmatrix} M_{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & M_{\lambda_p}\end{pmatrix} = np \quad \Rightarrow \quad \forall\lambda\in\sigma(A),\ \operatorname{rk} M_\lambda = n$
$$\Rightarrow \forall\lambda\in\sigma(A),\ \operatorname{rk}\left((A-\lambda I_n)^{n_\lambda} \quad (A-\lambda I_n)^{n_\lambda-1}B \quad \dots \quad (A-\lambda I_n)B \quad B\right) = n$$
$$\Rightarrow \forall\lambda\in\sigma(A),\ \ker(A^\top - \lambda I_n)^{n_\lambda} \cap \left(\bigcap_{k=0}^{n_\lambda-1} \ker B^\top (A^\top - \lambda I_n)^k\right) = \{0\}$$
$$\Rightarrow \forall\lambda\in\sigma(A),\ \ker(A^\top - \lambda I_n) \cap \ker B^\top = \{0\}$$
$$\Rightarrow \forall\lambda\in\sigma(A),\ \operatorname{rk}\left(A-\lambda I_n \quad B\right) = n.$$

We recover the classical Hautus test.

Hautus test for output controllability III

Proof

Proof.

3. We prove that our criterion is equivalent to $\operatorname{rk} C\begin{pmatrix}B & AB & \dots & A^{n-1}B\end{pmatrix} = q$.
Assume $\operatorname{rk} C = q$ (if $\operatorname{rk} C < q$, the proof is trivial).
   - Assume $\operatorname{Im} C^\top \cap \bigoplus_{\lambda\in\sigma(A)} E_\lambda \neq \{0\}$.
     There exist $\eta \in \mathbb{R}^q$ such that $C^\top \eta \neq 0$ and there exist $z_\lambda \in E_\lambda$ $(\forall \lambda \in \sigma(A))$ such that
     $$C^\top \eta = \sum_{\lambda\in\sigma(A)} z_\lambda.$$
     We then deduce, $B^\top {A^k}^\top C^\top \eta = \sum_{\lambda\in\sigma(A)} B^\top {A^k}^\top z_\lambda = 0.$
     That is to say, $\operatorname{rk} C\begin{pmatrix}B & AB & \dots & A^{n-1}B\end{pmatrix} < q$.
   - Reciprocally, if $\operatorname{rk} C\begin{pmatrix}B & AB & \dots & A^{n-1}B\end{pmatrix} < q$.
     There exists $\eta \in \mathbb{R}^q \backslash \{0\}$ such that $B^\top {A^k}^\top C^\top \eta = 0$ for all $k \in \mathbb{N}$.
     We deduce that $C^\top \eta \in \left(\operatorname{Im} C^\top \cap \mathcal{N}\right) \setminus \{0\}$ where $\mathcal{N} = \bigcap_{k\in\mathbb{N}} \ker B^\top (A^k)^\top$.
     Since $\mathcal{N}$ is $A^\top$ invariant, we have $\mathcal{N} = \bigoplus_{\lambda\in\sigma(A)} \left(\mathcal{N} \cap \ker \left(A^\top - \lambda I_n\right)^{n_\lambda}\right)$.
     We conclude by noticing that $\mathcal{N} \cap \ker \left(A^\top - \lambda I_n\right)^{n_\lambda} = E_\lambda$.
4. Simply observe that $E_\lambda = \ker M_\lambda^\top$, and conclude by duality. □

A simple observation

Given $x^0 \in \mathbb{R}^n$ and $T > 0$.

State controllability case

If at time T, we have $x(T) = 0$, taking $u(t) = 0$ for $t > T$ implies

$$x(t) = 0 \qquad (t > T).$$

Output controllability case

If at time T, we have $y(T) = 0$, taking $u(t) = 0$ for $t > T$ does not imply

$$y(t) = 0 \qquad (t > T).$$

This would automatically append if $\ker C$ is stable by A.

A slight modification of the output controllability notion

Definition

We say that the system (1) (still with $D = 0$) is *output controllable*, if for all $x^0 \in \mathbb{R}^n$ and all $y^1 \in \operatorname{Im} C$, these exits $u \in L^2([0, T], \mathbb{R}^m)$ such that $y(T, x^0, u) = y^1$.

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With this definition, we easily have the following necessary and sufficient condition.

Proposition

System (1) is output controllable if and only if

$$\operatorname{rk} C \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = \operatorname{rk} C.$$

The basic idea – necessary condition for long-time output controllability I

Of course, it is required that the system is output controllable.

- At time T, we have $x(T) = \bar{x}$, with $\bar{x} \in \mathbb{R}^n$ such that $C\bar{x} = y^1$.
  The aim is to find u such that $Cx(t) = y^1$ for every $t \geqslant T$.

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- If this holds, then, $C\dot{x}(t) = 0$, i.e., $CAx(t) + CBu(t) = 0$.

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- If this holds, then, $C\dot{x}(t) = 0$, i.e., $CAx(t) + CBu(t) = 0$.
- Let us define P the orthogonal projector of $\mathbb{R}^q$ on $\ker(CB)^\top$.
  We have $P = I_q - QCB(QCB)^\top$, where Q is the Gram-Schmidt matrix ensuring that $\operatorname{Im} QCB = \operatorname{Im} CB$, and the columns of QCB are orthonormal. We then have,

$$-PCAx(t) = PCBu(t) = 0.$$

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- If this holds, then, $C\dot{x}(t) = 0$, i.e., $CAx(t) + CBu(t) = 0$.
- Let us define P the orthogonal projector of $\mathbb{R}^q$ on $\ker(CB)^\top$,

$$-PCAx(t) = PCBu(t) = 0.$$

- In addition to $Cx(t) = y^1$, one has to satisfy,

$$\begin{pmatrix} C \\ PCA \end{pmatrix} x(t) = \begin{pmatrix} y^1 \\ 0 \end{pmatrix} = \begin{pmatrix} C \\ PCA \end{pmatrix} \bar{x}.$$

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- If long-time output controllability holds, it must also holds, with:

  C replaced by $C_1 = \begin{pmatrix} C \\ PCA \end{pmatrix}$, and y^1 replaced by $\begin{pmatrix} y^1 \\ 0 \end{pmatrix}$.

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- In addition to $Cx(t) = y^1$, one has to satisfy,
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- If long-time output controllability holds, it must also holds, with:
$$C \text{ replaced by } C_1 = \begin{pmatrix} C \\ PCA \end{pmatrix}, \text{ and } y^1 \text{ replaced by } \begin{pmatrix} y^1 \\ 0 \end{pmatrix}.$$
- Iterate, *until it stops*.

The basic idea – necessary condition for long-time output controllability II

Based on the previous considerations, we define the iterative sequence $(C_k)_{k\in\mathbb{N}}$,

- $C_0 = C$
- for $k \in \mathbb{N}$, we set $C_{k+1} = \begin{pmatrix} C \\ P_k C_k A \end{pmatrix} \in \mathbb{R}^{(k+1)q\times n}$,

  with P_k the orthogonal projector of $\mathbb{R}^{(k+1)q}$ on $\ker(C_k B)^\top$.

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  with P_k the orthogonal projector of $\mathbb{R}^{(k+1)q}$ on $\ker(C_k B)^\top$.

Lemma

We have,

1. $\ker C_{k+1} \subset \ker C_k \subset \mathbb{R}^n$;
2. *There exist $K \in \{0, \dots, n\}$ such that* $\ker C_{K+1} = \ker C_K$;
3. *For every $i \in \mathbb{N}$,* $\ker C_{K+i} = \ker C_K$.

The basic idea – necessary condition for long-time output controllability III

Proof.

1. It is clear that $\ker C_1 = \ker C_0 \cap \ker P_0 C_0 A \subseteq \ker C_0$.
   Suppose that $\ker C_k \subseteq \ker C_{k-1}$ for some $k \in \mathbb{N}^\top$. We have

$$\begin{aligned} P_k C_k A x = 0 \Leftrightarrow C_k(Ax - Bu) = 0, \quad \text{for some } u \in \mathbb{R}^m \\ \Rightarrow C_{k-1}(Ax - Bu) = 0 \Leftrightarrow P_{k-1} C_{k-1} A x = 0. \end{aligned}$$

   Thus, $\ker C_{k+1} \subseteq \ker C_k \subseteq \mathbb{R}^n$ for every $k \in \mathbb{N}$.

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   Thus, $\ker C_{k+1} \subseteq \ker C_k \subseteq \mathbb{R}^n$ for every $k \in \mathbb{N}$.

2. It is then easy to show the existence of $K \in \{0, \ldots, n\}$ such that $\ker C_{K+1} = \ker C_K$.

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   Suppose that $\ker C_k \subseteq \ker C_{k-1}$ for some $k \in \mathbb{N}^\top$. We have
   $$\begin{aligned} P_k C_k Ax = 0 &\Leftrightarrow C_k(Ax - Bu) = 0, \quad \text{for some } u \in \mathbb{R}^m \\ &\Rightarrow C_{k-1}(Ax - Bu) = 0 \Leftrightarrow P_{k-1} C_{k-1} Ax = 0. \end{aligned}$$
   Thus, $\ker C_{k+1} \subseteq \ker C_k \subseteq \mathbb{R}^n$ for every $k \in \mathbb{N}$.

2. It is then easy to show the existence of $K \in \{0, \ldots, n\}$ such that $\ker C_{K+1} = \ker C_K$.

3. From the structure of operators C_k, we have $\ker P_{K+1} C_{K+1} A = \ker P_K C_K A$, and hence $\ker C_{K+2} = \ker C_{K+1} \ldots$ □

Sufficient condition for long-time output controllability

Lemma

If $x(T) = \bar{x}$ *is such that* $C_K \bar{x} = \begin{pmatrix} y^1 \\ 0 \end{pmatrix}$, *then there exist* $u \in L^2_{loc}([T, \infty), \mathbb{R}^m)$ *such that the solution of* (1) *satisfies* $Cx(t) = y^1$ *for every* $t > T$.

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Proof. We have $C_K\dot{x} = P_K C_K Ax + (I_{(K+1)q} - P_K)C_K Ax + C_K Bu$.
Recall that P_K is the orthonormal projector on $\ker(C_K B)^\top$, hence,

$$(I_{(K+1)q} - P_K)C_K Ax \in \operatorname{Im} C_K B,$$

and we can choose $u = u(x)$ such that

$$(I_{(K+1)q} - P_K)C_K Ax = -C_K Bu.$$

With this choice, we have

$$C_K\dot{x} = P_K C_K Ax.$$

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Proof. With a good choice of u, we have

$$C_K \dot{x} = P_K C_K A x.$$

Let us write

$$x(t) = \bar{x} + x_0(t) + x_1(t), \quad \text{with } x_0(t) \in \ker C_K \text{ and } x_1(t) \in \operatorname{Im} C_K^\top.$$

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We have $C_K\dot{x} = C_K\dot{x}_1 = P_K C_K A\,(\bar{x} + x_0 + x_1)$.
By assumption, $\bar{x} \in \ker P_K C_K A$, and recall that $\ker C_K = \ker C_{K+1} = \ker C_0 \cap \ker P_K C_K A$, implying that $\ker C_K \subset \ker P_K C_K A$.
We deduce

$$C_K\dot{x}_1 = P_K C_K A x_1.$$

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If $x(T) = \bar{x}$ is such that $C_K \bar{x} = \begin{pmatrix} y^1 \\ 0 \end{pmatrix}$, then there exist $u \in L^2_{loc}([T, \infty), \mathbb{R}^m)$ such that the solution of (1) satisfies $Cx(t) = y^1$ for every $t > T$.

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Set $z = C_K x_1$.
$C_K : \operatorname{Im} C_K^\top \to \operatorname{Im} C_K$ is regular, hence,
there exist $\Theta : \operatorname{Im} C_K \to \operatorname{Im} C_K^\top$ such that $\Theta z = x_1$. Thus,

$$\dot{z} = P_K C_K A \Theta z.$$

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We deduce

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$z = C_K x_1$ and $x_1 = \Theta z$,

$$\dot{z} = P_K C_K A \Theta z.$$

This together with $x_1(T) = 0$ implies $z(T) = 0$, $z(t) = 0$ for every $t \geqslant T$, and finally,

$$C_K \dot{x} = 0.$$

□

Necessary and sufficient condition for long-time output controllability

Theorem

Given $y^1 \in \operatorname{Im} C$ *and* $T > 0$. *For every* $x^0 \in \mathbb{R}^n$ *there exists a control* $u \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ *such that the solution to the system* (1) *satisfies* $Cx(t) = y^1$ *for every* $t \geqslant T$ *if and only if*

$$\begin{pmatrix} y^1 \\ 0 \end{pmatrix} \in \operatorname{Im} C_K \quad \text{and} \quad \operatorname{rk} C_K \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = \operatorname{rk} C_K$$

Norm optimal control I

Given $T_0 > 0$, $T_1 > 0$, and $x^0 \in \mathbb{R}^n$, we aim to find the control minimizing:

$$\min \quad \frac{1}{2}\|u\|^2_{L^2(0,T_0+T_1)}$$
$$\Big| \quad Cx(t) = y^1 \qquad (t \in (T_0, T_0 + T_1)).$$

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Using Bellman principle, this minimum is:

$$\min \quad J_0(\bar{x}, T_0) + J_1(\bar{x}, T_1) \quad \Big| \quad \bar{x} \in \{e^{T_0 A} x^0\} + \operatorname{Im}\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}, \quad C_K \bar{x} = \begin{pmatrix} y^1 \\ 0 \end{pmatrix},$$

with

$$J_0(\bar{x}, T_0) = \min \tfrac{1}{2}\|u\|^2_{L^2([0,T_0],\mathbb{R}^m)} \quad \Big| \quad u \in L^2([0,T_0],\mathbb{R}^m), \quad x(T_0) = \bar{x}$$

$$J_1(\bar{x}, T_1) = \min \tfrac{1}{2}\|u\|^2_{L^2([0,T_1],\mathbb{R}^m)} \quad \Big| \quad u \in L^2([0,T_1],\mathbb{R}^m), \quad Cx(t) = C\bar{x} \ (t \in (0, T_1)).$$

Norm optimal control II

Lemma

There exist $E \in \mathbb{R}^{n\times n}$ (solution at time $t = 0$ of a backward Riccati equation set $(0, T_1)$) such that

$$J_1(\bar{x}, T_1) = -\bar{x}^\top E\, \bar{x}.$$

The original problem can hence be reset as:

$$\min \quad \frac{1}{2}\|u\|^2_{L^2(0,T_0)} - x(T_0)^\top E x(T_0)$$

$$\left| \begin{array}{l} u \in L^2([0, T_0], \mathbb{R}^m), \\ C_K x(T_0) = \begin{pmatrix} y^1 \\ 0 \end{pmatrix}. \end{array} \right.$$

Illustration I

Consider the system (1) with,

$$A = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad C = \frac{1}{2}\begin{pmatrix} I_2 & I_2 \end{pmatrix}$$

This can be seen as an *averaged controllability system*[6].

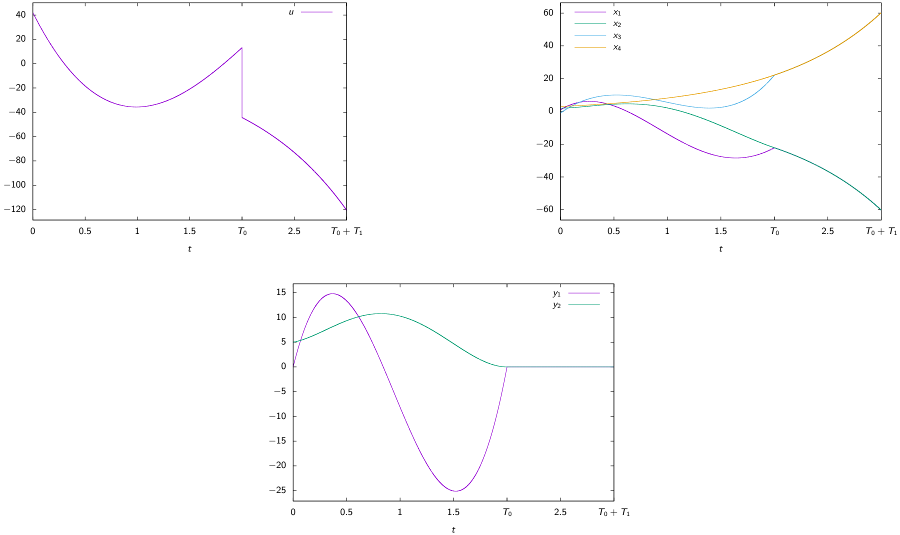
Taking $y^1 = 0$, one can check that for every $T > 0$, there exist a control $u \in L^2_{loc}(\mathbb{R}_+)$ such that $Cx(t) = y^1$ for every $t > T$.

We can also look for a control of minimal L^2-norm.

[6] E. Zuazua. "Averaged control". *Automatica J. IFAC* 50.12 (2014)

Illustration II

$$T_0 = 2,\ T_1 = 1 \text{ and } x^0 = \begin{pmatrix} 1 & 2 & -1 & 3 \end{pmatrix}^\top$$



Conclusion I

Note that we can design systems which are:

- not output controllable;
- output controllable, but not long-time output controllable;
- long time output controllable, but not state controllable;
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Comments:

- The proposed Hautus test, might be hard to use for checking the output controllability.
- Computing C_K might be hard in practice.

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- The proposed Hautus test, might be hard to use for checking the output controllability.
- Computing C_K might be hard in practice.

Open questions:

- What is the behavior of the minimal L^2-norm output control to 0 with respect to the control time T?
- What about feed-back output control?

Conclusion II

The presented results are taken from:

- B. Danhane, J. Lohéac, and M. Jungers. "Characterizations of output controllability for LTI systems". Preprint. 2020
- M. Lazar and J. Lohéac. "Output controllability in a long-time horizon". *Automatica J. IFAC* 113 (2020)

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Thank you for your attention!