Application of homogenization to an optimal design problem

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Setting the problem

- Let us consider mixtures of two conducting phases characterized by two symmetric positive definite tensors A_0 and A_1.
- We denote by η the amplitude or contrast between the two materials: $A_1 = A_0(1+\eta)$, $|\eta| \ll 1$
- Denoting by χ the characteristic function of the region occupied by phase A_1, we define a conductivity tensor $A = (1-\chi)A_0 + \chi A_1 = A_0(1+\eta\chi)$

$$\mathcal{U}_{ad1} = \left\{ \chi \in L^\infty(\Omega; \{0,1\}) : \int_\Omega \chi(x)dx = \Theta|\Omega| a.e. t \in \langle 0, T \rangle \right\} ,$$

or

$$\mathcal{U}_{ad2} = \left\{ \chi \in L^\infty(Q; \{0,1\}) : \int_Q \chi(t,x)dxdt = \Theta|Q| \right\} .$$

- $\mathcal{U}_{ad}$ is $\mathcal{U}_{ad1}$ or $\mathcal{U}_{ad2}$
    1. $V := H_0^1(\Omega)$, $V' := H^{-1}(\Omega)$ and $H := L^2(\Omega)$, $V \hookrightarrow H \hookrightarrow V'$
    2. $\mathcal{V} := L^2(0,T;V)$, $\mathcal{V}' := L^2(0,T;V')$ and $\mathcal{H} := L^2(0,T;H)$, $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$

- We want to minimize an objective function of the type:
  $J(\chi) = \int_0^T \int_\Omega j(u) dxdt$, $j \in C^3$ such that:
  $\exists C > 0$ $|j(u)| \leq C(|u|^2 + 1)$, $|j'(u)| \leq C(|u| + 1)$, $|j''(u)| \leq C$, where u is unique solution of problem:

$$\left\{ \begin{array}{c} \dot{u} - \mathrm{div}(A \nabla u) = f \text{ in } Q = \langle 0, T \rangle \times \Omega \\ u = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u(0, \cdot) = g \text{ in } \Omega \end{array} \right. \tag{1}$$

Problem 1

We call "orginal" optimal design problem

$$\inf_{\chi \in \mathcal{U}_{ad}} J(\chi) \,.$$

1st step

- Since the matrix A is an affine function of η, the solution u is analytic with respect to η, and we can write

$$u = u_0 + \eta u_1 + \eta^2 u_2 + \ldots \approx u_0 + \eta u_1 + \eta^2 u_2$$

- We make a Taylor expansion in the objective function to get:

$$\begin{aligned} J(\chi) &\approx \int_0^T \int_\Omega \left(j(u_0) + \eta j'(u_0) u_1 + \eta^2 \left(j'(u_0) u_2 + \frac{1}{2} j''(u_0)(u_1)^2 \right) \right) \\ &=: J_{sa}(\chi)\,. \end{aligned}$$

- Plugging this in (1) yields three equations for (u_0, u_1, u_2)

$$\begin{cases} \dot{u_0} - \operatorname{div}(A_0 \nabla u_0) = f \text{ in } Q \\ u_0 = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_0(0, \cdot) = g \text{ in } \Omega \end{cases} \tag{2}$$

$$\begin{cases} \dot{u_1} - \operatorname{div}(A_0 \nabla u_1) = \operatorname{div}(\chi A_0 \nabla u_0) \text{ in } Q \\ u_1 = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_1(0, \cdot) = 0 \text{ in } \Omega \end{cases} \tag{3}$$

$$\begin{cases} \dot{u_2} - \operatorname{div}(A_0 \nabla u_2) = \operatorname{div}(\chi A_0 \nabla u_1) \text{ in } Q \\ u_2 = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_2(0, \cdot) = 0 \text{ in } \Omega \end{cases} \tag{4}$$

Problem 2

We call "small amplitude" optimal design problem the second-order asymptotic of problem (P), namely

$$\inf_{\chi \in \mathcal{U}_{ad}} J_{sa}(\chi) \tag{5}$$

2nd step

- Instead of one characteristic function we take a sequence of characteristic functions
- Instead of solutions u_1, u_2 we take sequences of solutions associated to the sequence of characteristic functions

$$\begin{cases} \dot{u_0} - \operatorname{div}(A_0 \nabla u_0) = f \text{ in } Q \\ u_0 = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_0(0, x) = g \text{ in } \Omega \end{cases}$$

$$\begin{cases} \dot{u_1^n} - \operatorname{div}(A_0 \nabla u_1^n) = \operatorname{div}(\chi^n A_0 \nabla u_0) \text{ in } Q \\ u_1^n = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_1^n(0, x) = 0 \text{ in } \Omega \end{cases}$$

$$\begin{cases} \dot{u_2^n} - \operatorname{div}(A_0 \nabla u_2^n) = \operatorname{div}(\chi^n A_0 \nabla u_1^n) \text{ in } Q \\ u_2^n = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_2^n(0, x) = 0 \text{ in } \Omega \end{cases}$$

$$\begin{cases} \dot{u}_1^n - \operatorname{div}(A_0 \nabla u_1^n) = \operatorname{div}(\chi^n A_0 \nabla u_0) \text{ in } Q \\ u_1^n = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_1^n(0, \cdot) = 0 \text{ in } \Omega \end{cases}$$

$$\left\{ \begin{array}{c} \dot{u}_1 - \mathrm{div}(A_0 \nabla u_1) = \mathrm{div}(\theta A_0 \nabla u_0) \text{ in } Q \\ u_1 = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_1(0, \cdot) = 0 \text{ in } \Omega \end{array} \right. \tag{6}$$

$$\begin{cases} \dot{u}_2^n - \mathrm{div}(A_0 \nabla u_2^n) = \mathrm{div}(\chi^n A_0 \nabla u_1^n) \text{ in } Q \\ u_2^n = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_2^n(0, x) = 0 \text{ in } \Omega \end{cases}$$

$$\begin{aligned} \dot{u}_1^n - \mathrm{div}(A_0 \nabla u_1^n) &= \mathrm{div}(\chi^n A_0 \nabla u_0) \\ \mathcal{F}(\nabla u_1^n) &= -\frac{(2\pi)^2 (\xi \otimes \xi) \mathcal{F}(\chi^n \nabla u_0)}{-2\pi i \tau + (2\pi)^2 A_0 \xi \cdot \xi} \end{aligned}$$

Antonić, Lazar

If u^n is a sequence of functions in $L^2(\mathbb{R}^{1+N};\mathbb{R}^r)$ such that $u^n \rightharpoonup 0$ (weakly), then there exists a subsequence $(u^{n'})$ and a complex $r \times r$ matrix Radon measure μ on $\mathbb{R}^{1+N} \times P^N$ such that for any $\phi_1, \phi_2 \in C_0(\mathbb{R}^{1+N})$ i $\psi \in C(P^N)$:

$$\begin{aligned} \lim_{n'} \int_{\mathbb{R}^{1+N}} \mathcal{F}(\phi_1 u^{n'}) \otimes \mathcal{F}(\phi_2 u^{n'})(\psi \circ \pi)\, d\xi &= \langle \mu, (\phi_1 \bar{\phi}_2) \boxtimes \psi \rangle \\ &= \int_{\mathbb{R}^{1+N} \times P^N} \phi_1(x) \bar{\phi}_2(x) \psi(\xi)\, d\mu(x,\xi)\,. \end{aligned}$$

Measure μ from the above theorem we call (parabolic) variant H-measure associated to (a sub)sequence (of) u^n.

$P^N \ldots (2\pi\tau)^2 + (2\pi\xi)^4 = 1$

$\pi(\tau,\xi) = \left(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\right)$

$$\begin{cases} \dot{u}_2^n - \operatorname{div}(A_0 \nabla u_2^n) = \operatorname{div}(\chi^n A_0 \nabla u_1^n) \text{ in } Q \\ u_2^n = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_2^n(0, x) = 0 \text{ in } \Omega \end{cases}$$

$$\begin{cases} \dot{u}_2 - \operatorname{div}(A_0 \nabla u_2) = \operatorname{div}(\theta A_0 \nabla u_1) - \operatorname{div}(\theta(1-\theta) A_0 M A_0 \nabla u_0) \text{ in } Q \\ u_2 = 0 \text{ on } \langle 0, T \rangle \times \partial\Omega \\ u_2(0, \cdot) = 0 \text{ in } \Omega \end{cases} \tag{7}$$

where $M(t,x) = \int_{P^N} \frac{(2\pi)^2 \xi \otimes \xi}{-2\pi i \tau + (2\pi)^2 A_0 \xi \cdot \xi} \nu(t, x, d\tau, d\xi)$.

$$\lim_{n\to\infty} J_{sa}(\chi) = J^*_{sa}(\theta,\nu)\,,$$

where u_0, u_1 i u_2 are solutions of the state equations (2), (6) and (7) respectively.

Problem 3

We call "relaxed small amplitude" optimal design problem

$$\inf_{(\theta,\nu)\in\mathcal{U}^*_{ad}} J^*_{sa}(\theta,\nu)\,, \tag{8}$$

where

$$\mathcal{U}^*_{ad} = \left\{ \begin{array}{c} (\theta,\nu) \in L^\infty(Q;[0,1]) \times \mathcal{P}(Q,P^N) \\ t.d.\;\; \int_Q \theta(t,x)dtdx = \Theta|Q| \end{array} \right\}$$

$\int_{P^N} \nu(t,x,\tau,\xi)d\tau d\xi = 1 \; a.e \; (t,x) \in Q$
$P^N \ldots (2\pi\tau)^2 + (2\pi\xi)^4 = 1$

Teorem 1

(P3) is relaxation of problem (P2) in sense that exist $(\theta^*, \nu^*) \in \mathcal{U}_{ad}^*$ *such that*

$$J_{sa}^*(\theta^*, \nu^*) = \inf_{(\theta,\nu)\in\mathcal{U}_{ad}^*} J_{sa}^*(\theta, \nu) \tag{9}$$

and

$$\inf_{\chi\in\mathcal{U}_{ad}} J_{sa}(\chi) = \inf_{(\theta,\nu)\in\mathcal{U}_{ad}^*} J_{sa}^*(\theta, \nu)\,. \tag{10}$$

Propozicija 2

$$\min_{(\theta,\nu)\in\mathcal{U}_{ad}^{sl}} J_{sa}^*(\theta,\nu) = \min_{(\theta,\nu)\in\mathcal{U}_{ad}^{*}} J_{sa}^*(\theta,\nu) \tag{11}$$

where $\mathcal{U}_{ad}^{sl} \subset \mathcal{U}_{ad}^{*}$ *is*

$$\mathcal{U}_{ad}^{sl} = \left\{ \begin{array}{c} (\theta,\nu) \in L^\infty(Q;[0,1]\times\mathcal{P}(Q,P^N)) \text{ such that} \\ \int_Q \theta(t,x)dtdx = \Theta|Q|, \\ \nu(t,x,\tau,\xi) = \delta(\tau-\tau^*(t,x),\xi-\xi^*(t,x)) \text{ a.e. } (t,x)\in Q \end{array} \right\}$$

$$\begin{aligned} J^*_{sa}(\theta) &= \int_Q j(u_0)dtdx - \eta \int_Q \theta A_0 \nabla u_0 \cdot \nabla p_0 dtdx \\ &+ \eta^2 \int_Q \frac{1}{2} j''(u_0)(u_1)^2 dtdx - \eta^2 \int_Q \theta A_0 \nabla u_1 \cdot \nabla p_0 dtdx \\ &+ \eta^2 \int_Q \theta(1-\theta) A_0 M^* A_0 \nabla u_0 \cdot \nabla p_0 dtdx , \end{aligned}$$

where $M^* = \dfrac{(2\pi)^2 \xi^* \otimes \xi^*}{-2\pi i \tau^* + (2\pi)^2 A_0 \xi^* \cdot \xi^*}$.

$$\begin{cases} \dot{p}_0 + \operatorname{div}(A_0 \nabla p_0) = -j'(u_0) \text{ in } Q \\ p_0 = 0 \text{ on } \langle 0, T\rangle \times \partial\Omega \\ p_0(T, \cdot) = 0 \text{ in } \Omega \end{cases}$$

Lemma 3

The objective function $J^*_{sa}(\theta)$ is Frećhet differentiable and its derivative in the direction $s \in L^\infty(Q)$ is given by

$$\begin{aligned}\frac{\partial J^*_{sa}}{\partial \theta}(s) &= -\eta \int_Q sA_0 \nabla u_0 \cdot \nabla p_0 dtdx - \eta^2 \int_Q sA_0 \nabla u_1 \cdot \nabla p_0 dtdx \\ &+ \eta^2 \int_Q s(1-2\theta) A_0 M^* A_0 \nabla u_0 \cdot \nabla p_0 dtdx \\ &- \eta^2 \int_Q sA_0 \nabla u_0 \cdot \nabla p_1 dtdx \,,\end{aligned}$$

where p_1 is another adjoint state:

$$\left\{ \begin{array}{c} \dot{p}_1 + div(A_0 \nabla p_1) = j''(u_0) u_1 - div(\theta A_0 \nabla p_0) \ in \ Q \\ p_1 = 0 \ on \ \langle 0, T \rangle \times \partial\Omega \\ p_1(T, \cdot) = 0 \ in \ \Omega \,. \end{array} \right.$$

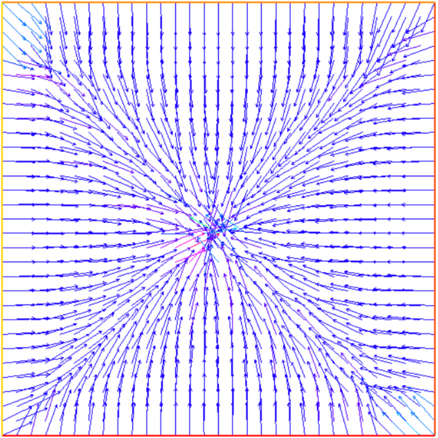
Problem

Find optimal arrangement of materials:

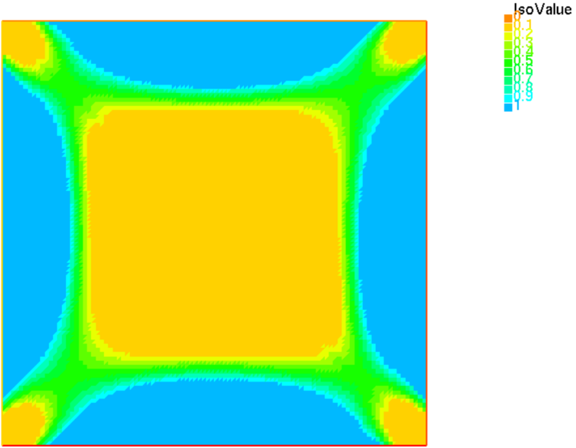
$$J(\chi) = \int_Q -u(t,x)dxdt\,,$$

$$\begin{cases} \dot{u} - \mathrm{div}(2\mathbb{I}(1-0.5\chi)\nabla u) = 1 \text{ in } Q = \langle 0,1\rangle \times \langle 0,1\rangle^2 \\ u = 0 \text{ on } \langle 0, T\rangle \times \partial\Omega \\ u(0,\cdot) = 0 \text{ in } \Omega\,. \end{cases}$$

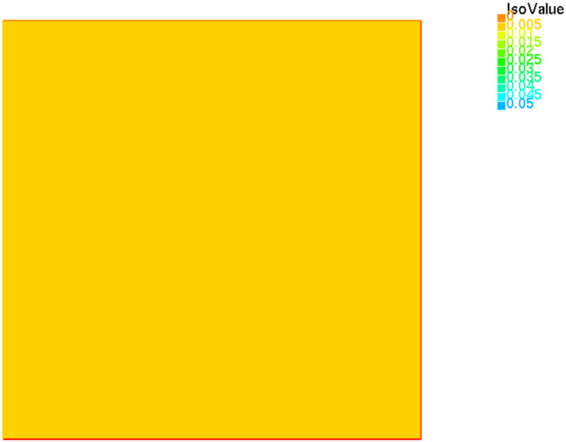
ξ^*



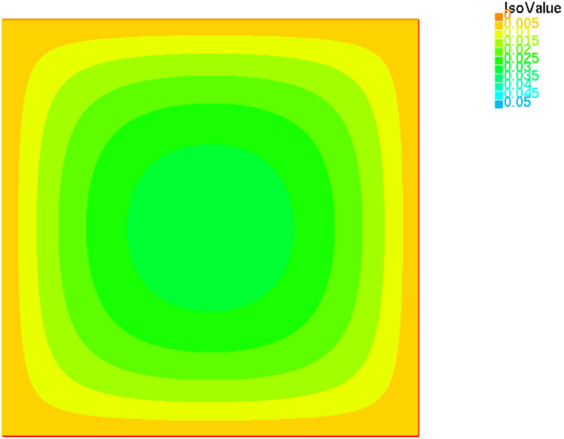
θ



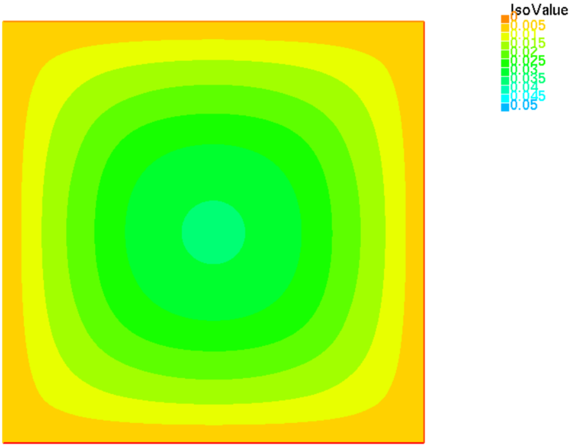
$u(0, \cdot)$



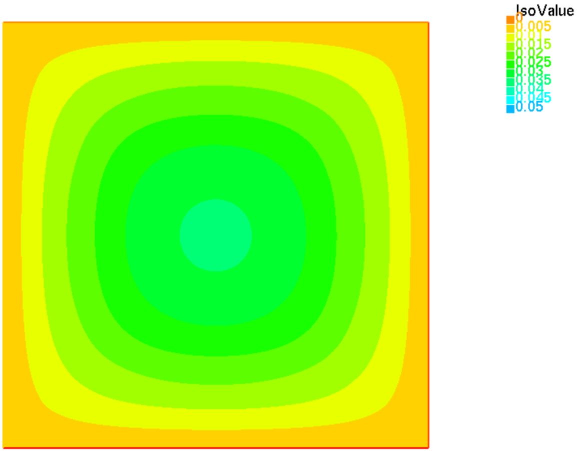
$u(0.1, \cdot)$



$u(0.5, \cdot)$



$u(1, \cdot)$



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Thank you!