Two-phase optimal design for elastic plate

Jelena Jankov

J. J. STROSSMAYER UNIVERSITY OF OSIJEK
DEPARTMENT OF MATHEMATICS
Trg Ljudevita Gaja 6
31000 Osijek, Hrvatska
http://www.mathos.unios.hr

jjankov@mathos.hr

Joint work with **Krešimir Burazin, Ivana Crnjac**

Croatian Science Foundation (IP8904)

[WORKSHOP ON CONTROL OF DYNAMICAL SYSTEMS, DUBROVNIK]

June, 2021

Kirchhoff-Love plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega). \end{cases}$$

P. G. Ciarlet, *Mathematical Elasticity, volume II: Theory of Plates,* Elsevier Science, Amsterdam, 1997.

- $\Omega \subseteq \mathbb{R}^d$ bounded domain ($d = 2 \ldots$ plate)
- $f \in H^{-2}(\Omega)$ external load
- $u \in H_0^2(\Omega)$ vertical displacement of the plate
- $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{\mathbf{N} \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\, \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \alpha \mathbf{S} : \mathbf{S} \text{ and } \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \frac{1}{\beta}\mathbf{S} : \mathbf{S} \text{ a.e. } \mathbf{x}\}$ describes properties of material of the given plate

Kirchhoff-Love plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega). \end{cases}$$

P. G. Ciarlet, *Mathematical Elasticity, volume II: Theory of Plates,* Elsevier Science, Amsterdam, 1997.

- $\Omega \subseteq \mathbb{R}^d$ bounded domain ($d = 2 \ldots$ plate)
- $f \in H^{-2}(\Omega)$ external load
- $u \in H_0^2(\Omega)$ vertical displacement of the plate
- $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{\mathbf{N} \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\, \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \alpha \mathbf{S} : \mathbf{S} \text{ and } \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \frac{1}{\beta}\mathbf{S} : \mathbf{S} \text{ a.e. } \mathbf{x}\}$ describes properties of material of the given plate

Antonić, Balenović, 1999.

Definition

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \end{cases}$$

coverges weakly to a limit u in $H_0^2(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $L^2(\Omega; \mathrm{Sym})$.

Theorem

Let $(\mathbf{M}^n)$ be a sequence in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. Then there is a subsequence $(\mathbf{M}^{n_k})$ and a tensor function $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ such that $(\mathbf{M}^{n_k})$ H-converges to $\mathbf{M}$.

Antonić, Balenović, 1999.

Definition

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \end{cases}$$

coverges weakly to a limit u in $H_0^2(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $L^2(\Omega; \mathrm{Sym})$.

Theorem

Let $(\mathbf{M}^n)$ be a sequence in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. Then there is a subsequence $(\mathbf{M}^{n_k})$ and a tensor function $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ such that $(\mathbf{M}^{n_k})$ H-converges to $\mathbf{M}$.

Properties

K. Burazin, J. Jankov, M. Vrdoljak, *Homogenization of elastic plate equation*, Mathematical Modelling and Analysis 23(2),190-204, 2018.

- Locality of the H-convergence
- Irrelevance of boundary conditions
- Energy convergence
- Ordering property
- Metrizability
- Corrector results

Definition

Let $\chi^n \in \mathrm{L}^\infty(\Omega; \{0,1\})$ *be a sequence of characteristic functions and* $(\mathbf{M}^n)$ *be a sequence of tensors defined by*

$$\mathbf{M}^n(\mathbf{x}) = \chi^n(\mathbf{x})\mathbf{A} + (1 - \chi^n(\mathbf{x}))\mathbf{B},$$

where **A** *and* **B** *are assumed to be positive definite fourth order tensors. Assume that there exist* $\theta \in \mathrm{L}^\infty(\Omega; [0,1])$ *and* $\mathbf{M} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ *such that*

$$\chi^n \xrightarrow{*} \theta \text{ in } \mathrm{L}^\infty(\Omega; [0,1]),$$
$$\mathbf{M}^n \xrightarrow{H} \mathbf{M} \text{ in } \mathfrak{M}_2(\alpha, \beta; \Omega).$$

The H-limit **M** *is said to be the homogenized tensor of a two-phase composite material obtained by mixing* **A** *and* **B** *in proportions* θ *and* $(1-\theta)$, *respectively, with a microstructure defined by the sequence* (χ^n).

Ω - mixture of two isotropic elastic phases with stiffness tensors

$$\mathbf{A} = 2\mu_1 \mathbf{I}_4 + (\kappa_1 - \mu_1)\mathbf{I}_2 \otimes \mathbf{I}_2$$
$$\mathbf{B} = 2\mu_2 \mathbf{I}_4 + (\kappa_2 - \mu_2)\mathbf{I}_2 \otimes \mathbf{I}_2$$

$$0 < \mu_1 \leq \mu_2,\ 0 < \kappa_1 \leq \kappa_2$$

$$\mathbf{M}(\mathbf{x}) = \chi(\mathbf{x})\mathbf{A} + (1 - \chi(\mathbf{x}))\mathbf{B}, \quad \chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$$

Ω - mixture of two isotropic elastic phases with stiffness tensors

$$\mathbf{A} = 2\mu_1 \mathbf{I}_4 + (\kappa_1 - \mu_1)\mathbf{I}_2 \otimes \mathbf{I}_2$$
$$\mathbf{B} = 2\mu_2 \mathbf{I}_4 + (\kappa_2 - \mu_2)\mathbf{I}_2 \otimes \mathbf{I}_2$$

$$0 < \mu_1 \leq \mu_2,\ 0 < \kappa_1 \leq \kappa_2$$

$$\mathbf{M}(\mathbf{x}) = \chi(\mathbf{x})\mathbf{A} + (1 - \chi(\mathbf{x}))\mathbf{B}, \quad \chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$$

Optimal design problem

$$\begin{cases} J(\chi) = \int_\Omega [\chi(\mathbf{x}) g_1(\mathbf{x}, u(\mathbf{x})) + (1-\chi(\mathbf{x})) g_2(\mathbf{x}, u(\mathbf{x}))] \, d\mathbf{x} \longrightarrow \min \,, \\ \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \,, \; \int_\Omega \chi \, d\mathbf{x} = q \,. \end{cases}$$

$0 < q < |\Omega|$, and g_1, g_2 Caratheodory functions which satisfy growth condition

$$g_{1,2}(\mathbf{x}, u) \leq a|u|^s + b(\mathbf{x}),$$

for some $a > 0$, $b \in \mathrm{L}^1(\Omega)$ and $1 \leq s < \frac{2d}{d-2}$, and u is the solution of

$$\begin{cases} \operatorname{div} \operatorname{div} (\mathbf{M} \nabla \nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega), \end{cases}$$

where $f \in \mathrm{H}^{-2}(\Omega)$, $\mathbf{M}(\mathbf{x}) = \chi(\mathbf{x})\mathbf{A} + (1-\chi(\mathbf{x}))\mathbf{B}$.

$$\begin{cases} J(\theta,\mathbf{M}) := \int_\Omega \left[\theta(\mathbf{x}) g_1(\mathbf{x},\mathsf{u}(\mathbf{x})) + (1-\theta)(\mathbf{x})) g_2(\mathbf{x},\mathsf{u}(\mathbf{x}))\right] d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x} \longrightarrow \min, \\ (\theta,\mathbf{M}) \in \mathrm{L}^\infty(\Omega;[0,1]\times\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))\,;\ \mathbf{M}(\mathbf{x}) \in G_{\theta(\mathbf{x})} \text{ a.e. on } \Omega. \end{cases} \tag{3.1}$$

Set of all composites:

$$\mathcal{A} := \{(\theta,\mathbf{M}) \in \mathrm{L}^\infty(\Omega;[0,1]\times\mathcal{L}(\mathrm{Sym},\mathrm{Sym})) :\ \mathbf{M}(\mathbf{x}) \in G_{\theta(\mathbf{x})} \text{ a.e. on } \Omega\}$$

G–closure problem: for given θ find all possible homogenized (effective) tensors **M** - open problem
For $g_1 = g_2 = fu$, it follows

$$J(\theta,\mathbf{M}) = \int_\Omega f(\mathbf{x})u(\mathbf{x})\, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x}.$$

$$\begin{cases} J(\theta,\mathbf{M}) := \int_\Omega [\theta(\mathbf{x}) g_1(\mathbf{x},\mathsf{u}(\mathbf{x})) + (1-\theta)(\mathbf{x})) g_2(\mathbf{x},\mathsf{u}(\mathbf{x}))]\, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x} \longrightarrow \min, \\ (\theta,\mathbf{M}) \in \mathrm{L}^\infty(\Omega;[0,1] \times \mathcal{L}(\mathrm{Sym},\mathrm{Sym}))\,;\ \mathbf{M}(\mathbf{x}) \in G_{\theta(\mathbf{x})} \text{ a.e. on } \Omega. \end{cases} \tag{3.1}$$

Set of all composites:

$$\mathcal{A} := \{(\theta,\mathbf{M}) \in \mathrm{L}^\infty(\Omega;[0,1] \times \mathcal{L}(\mathrm{Sym},\mathrm{Sym}))\ :\ \mathbf{M}(\mathbf{x}) \in G_{\theta(\mathbf{x})} \text{ a.e. on } \Omega\}$$

G–closure problem: for given θ find all possible homogenized (effective) tensors **M** - open problem

For $g_1 = g_2 = fu$, it follows

$$J(\theta,\mathbf{M}) = \int_\Omega f(\mathbf{x}) u(\mathbf{x})\, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x}.$$

$$\begin{cases} J(\theta,\mathbf{M}) := \int_\Omega \left[\theta(\mathbf{x}) g_1(\mathbf{x},\mathsf{u}(\mathbf{x})) + (1-\theta)(\mathbf{x})) g_2(\mathbf{x},\mathsf{u}(\mathbf{x}))\right] d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x} \longrightarrow \min, \\ (\theta,\mathbf{M}) \in \mathrm{L}^\infty\left(\Omega;[0,1]\times\mathcal{L}(\mathrm{Sym},\mathrm{Sym})\right);\ \mathbf{M}(\mathbf{x}) \in G_{\theta(\mathbf{x})} \text{ a.e. on } \Omega. \end{cases} \tag{3.1}$$

Set of all composites:

$$\mathcal{A} := \{(\theta,\mathbf{M}) \in \mathrm{L}^\infty(\Omega;[0,1]\times\mathcal{L}(\mathrm{Sym},\mathrm{Sym})) :\ \mathbf{M}(\mathbf{x}) \in G_{\theta(\mathbf{x})} \text{ a.e. on } \Omega\}$$

G–closure problem: for given θ find all possible homogenized (effective) tensors **M** - open problem

For $g_1 = g_2 = fu$, it follows

$$J(\theta,\mathbf{M}) = \int_\Omega f(\mathbf{x})u(\mathbf{x})\, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x}.$$

Homogenization of laminated structures

Theorem

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be two constant tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *and* $\chi_n(\mathbf{x} \cdot \mathbf{e})$ *be a sequence of characteristic functions that converges to* $\theta(\mathbf{x} \cdot \mathbf{e})$ *in* $\mathrm{L}^\infty(\Omega; [0,1])$ *weakly-*$*$*. Then, sequence* $(\mathbf{M}^n)$ *of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, defined as*

$$\mathbf{M}^n(\mathbf{x} \cdot \mathbf{e}) = \chi_n(\mathbf{x} \cdot \mathbf{e})\mathbf{A} + (1 - \chi_n(\mathbf{x} \cdot \mathbf{e}))\mathbf{B}$$

H-converges to

$$\mathbf{M} = \theta\mathbf{A} + (1-\theta)\mathbf{B} - \frac{\theta(1-\theta)(\mathbf{A}-\mathbf{B})(\mathbf{e}\otimes\mathbf{e}) \otimes (\mathbf{A}-\mathbf{B})^T(\mathbf{e}\otimes\mathbf{e})}{(1-\theta)\mathbf{A}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e}) + \theta\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}, \tag{3.2}$$

which also depends only on $\mathbf{x} \cdot \mathbf{e}$*.*

Corollary

If $(\mathbf{A} - \mathbf{B})$ *is an invertible fourth order tensor, formula (3.2) is equivalent to*

$$\theta(\mathbf{M} - \mathbf{B})^{-1} = (\mathbf{A} - \mathbf{B})^{-1} + \frac{1-\theta}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}). \quad (3.3)$$

If we repeat iterative process of lamination p times, in lamination directions $(\mathrm{e}_i)_{1\le i\le p}$ and proportions $(\theta_i)_{1\le i\le p}$, we obtain a rank-p sequential laminate with matrix **B** and core **A**, which is defined by the following formula:

$$\left(\prod_{j=1}^{p}\theta_j\right)(\mathbf{A}_p^{*}-\mathbf{B})^{-1} = (\mathbf{A}-\mathbf{B})^{-1}+\sum_{i=1}^{p}\left((1-\theta_i)\prod_{j=1}^{i-1}\theta_j\right)\frac{(\mathrm{e}_i\otimes\mathrm{e}_i)\otimes(\mathrm{e}_i\otimes\mathrm{e}_i)}{\mathbf{B}(\mathrm{e}_i\otimes\mathrm{e}_i):(\mathrm{e}_i\otimes\mathrm{e}_i)}.$$

$$\theta(\mathbf{M}^{-1} - \mathbf{B}^{-1})^{-1} = (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1} + (1-\theta)\left[\mathbf{B} - \mathbf{B}\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}\mathbf{B}\right]$$

Corollary

If $(\mathbf{A} - \mathbf{B})$ *is an invertible fourth order tensor, formula (3.2) is equivalent to*

$$\theta(\mathbf{M} - \mathbf{B})^{-1} = (\mathbf{A} - \mathbf{B})^{-1} + \frac{1-\theta}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}). \quad (3.3)$$

If we repeat iterative process of lamination p times, in lamination directions $(\mathbf{e}_i)_{1\le i\le p}$ and proportions $(\theta_i)_{1\le i\le p}$, we obtain a rank-p sequential laminate with matrix **B** and core **A**, which is defined by the following formula:

$$\left(\prod_{j=1}^{p}\theta_j\right)(\mathbf{A}_p^* - \mathbf{B})^{-1} = (\mathbf{A}-\mathbf{B})^{-1} + \sum_{i=1}^{p}\left((1-\theta_i)\prod_{j=1}^{i-1}\theta_j\right)\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i) : (\mathbf{e}_i\otimes\mathbf{e}_i)}.$$

$$\theta(\mathbf{M}^{-1} - \mathbf{B}^{-1})^{-1} = (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1} + (1-\theta)\left[\mathbf{B} - \mathbf{B}\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}) : (\mathbf{e}\otimes\mathbf{e})}\mathbf{B}\right]$$

Corollary

If $(\mathbf{A}-\mathbf{B})$ *is an invertible fourth order tensor, formula (3.2) is equivalent to*

$$\theta(\mathbf{M}-\mathbf{B})^{-1}=(\mathbf{A}-\mathbf{B})^{-1}+\frac{1-\theta}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}). \quad (3.3)$$

If we repeat iterative process of lamination p times, in lamination directions $(\mathbf{e}_i)_{1\le i\le p}$ and proportions $(\theta_i)_{1\le i\le p}$, we obtain a rank-p sequential laminate with matrix **B** and core **A**, which is defined by the following formula:

$$\left(\prod_{j=1}^{p}\theta_j\right)(\mathbf{A}_p^{*}-\mathbf{B})^{-1}=(\mathbf{A}-\mathbf{B})^{-1}+\sum_{i=1}^{p}\left((1-\theta_i)\prod_{j=1}^{i-1}\theta_j\right)\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}.$$

$$\theta(\mathbf{M}^{-1}-\mathbf{B}^{-1})^{-1}=(\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}+(1-\theta)\left[\mathbf{B}-\mathbf{B}\frac{(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}\mathbf{B}\right]$$

Lemma

Let θ be a volume fraction in $[0,1]$, $(\mathbf{e}_i)_{1\le i\le p}$ unit vectors in $\mathbf{R}^d$, and $(m_i)_{1\le i\le p}$ nonnegative real numbers such that $\sum_{i=1}^{p} m_i = 1$. Then, there exists a rank-p sequential laminate $\mathbf{A}_p^*$ with core $\mathbf{A}$ and matrix $\mathbf{B}$, and with lamination directions $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_p$, such that

$$\theta(\mathbf{A}_p^{*-1}-\mathbf{B}^{-1})^{-1} = (\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}+(1-\theta)\left[\mathbf{B}-\sum_{i=1}^{p} m_i \mathbf{B}\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}\mathbf{B}\right]. \tag{3.4}$$

One could interchange the roles of **A** and **B** and obtain a symmetric class of sequential laminates:

Lemma

Let θ be a volume fraction in $[0,1]$, $(\mathbf{e}_i)_{1\le i\le p}$ unit vectors in $\mathbf{R}^d$, $(m_i)_{1\le i\le p}$ nonnegative real numbers such that $\sum_{i=1}^{p} m_i = 1$. Then, there exists a rank-p sequential laminate $\mathbf{A}_p^*$ with core $\mathbf{B}$ and matrix $\mathbf{A}$, and with lamination directions $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_p$, such that

$$(1-\theta)(\mathbf{A}_p^{*-1}-\mathbf{A}^{-1})^{-1} = (\mathbf{B}^{-1}-\mathbf{A}^{-1})^{-1}+\theta\left[\mathbf{A}-\sum_{i=1}^{p} m_i \mathbf{A}\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}\mathbf{A}\right]. \tag{3.5}$$

Lemma

Let θ be a volume fraction in $[0,1]$, $(\mathbf{e}_i)_{1\le i\le p}$ unit vectors in $\mathbf{R}^d$, and $(m_i)_{1\le i\le p}$ nonnegative real numbers such that $\sum_{i=1}^{p} m_i = 1$. Then, there exists a rank-p sequential laminate $\mathbf{A}_p^*$ with core $\mathbf{A}$ and matrix $\mathbf{B}$, and with lamination directions $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_p$, such that

$$\theta(\mathbf{A}_p^{*-1}-\mathbf{B}^{-1})^{-1} = (\mathbf{A}^{-1}-\mathbf{B}^{-1})^{-1}+(1-\theta)\left[\mathbf{B}-\sum_{i=1}^{p} m_i \mathbf{B}\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{B}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}\mathbf{B}\right]. \tag{3.4}$$

One could interchange the roles of **A** and **B** and obtain a symmetric class of sequential laminates:

Lemma

Let θ be a volume fraction in $[0,1]$, $(\mathbf{e}_i)_{1\le i\le p}$ unit vectors in $\mathbf{R}^d$, $(m_i)_{1\le i\le p}$ nonnegative real numbers such that $\sum_{i=1}^{p} m_i = 1$. Then, there exists a rank-p sequential laminate $\mathbf{A}_p^*$ with core $\mathbf{B}$ and matrix $\mathbf{A}$, and with lamination directions $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_p$, such that

$$(1-\theta)(\mathbf{A}_p^{*-1}-\mathbf{A}^{-1})^{-1} = (\mathbf{B}^{-1}-\mathbf{A}^{-1})^{-1}+\theta\left[\mathbf{A}-\sum_{i=1}^{p} m_i \mathbf{A}\frac{(\mathbf{e}_i\otimes\mathbf{e}_i)\otimes(\mathbf{e}_i\otimes\mathbf{e}_i)}{\mathbf{A}(\mathbf{e}_i\otimes\mathbf{e}_i):(\mathbf{e}_i\otimes\mathbf{e}_i)}\mathbf{A}\right]. \tag{3.5}$$

Theorem

The homogenized tensor $\mathbf{A}^* \in G_\theta$ *satisfies*

$$(\forall \boldsymbol{\sigma} \in \mathrm{Sym}) \;\; \mathbf{A}^{*-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \mathbf{B}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \theta \max_{\boldsymbol{\eta} \in \mathrm{Sym}} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} - (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta) g^c(\boldsymbol{\eta}) \right], \tag{3.6}$$

where

$$g^c(\boldsymbol{\eta}) := \mathbf{B}\boldsymbol{\eta} : \boldsymbol{\eta} - h_{\mathbf{B}}(\boldsymbol{\eta}), \tag{3.7}$$

while $h_{\mathbf{B}}(\boldsymbol{\eta})$ *is defined with*

$$h_{\mathbf{B}}(\boldsymbol{\eta}) := \min_{\mathbf{e} \in S^{d-1}} \frac{|(\mathbf{e} \otimes \mathbf{e}) : \mathbf{B}\boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}. \tag{3.8}$$

Moreover,

$$(\forall \boldsymbol{\sigma} \in \mathrm{Sym}) \;\; \mathbf{A}^{*-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \leq \mathbf{A}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + (1-\theta) \min_{\boldsymbol{\eta} \in \mathrm{Sym}} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} + (\mathbf{A}^{-1} - \mathbf{B}^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - \theta h^c(\boldsymbol{\eta}) \right], \tag{3.9}$$

where

$$h^c(\boldsymbol{\eta}) := \mathbf{A}\boldsymbol{\eta} : \boldsymbol{\eta} - g_{\mathbf{A}}(\boldsymbol{\eta}), \tag{3.10}$$

while $g_{\mathbf{A}}(\boldsymbol{\eta})$ *is defined with*

$$g_{\mathbf{A}}(\boldsymbol{\eta}) := \max_{\mathbf{e} \in S^{d-1}} \frac{|(\mathbf{e} \otimes \mathbf{e}) : \mathbf{A}\boldsymbol{\eta}|^2}{\mathbf{A}(\mathbf{e} \otimes \mathbf{e}) : (\mathbf{e} \otimes \mathbf{e})}. \tag{3.11}$$

Additionally, (3.6) and (3.9) are optimal and optimality is achieved by a rank-p *sequential laminate, where* $p = \dfrac{(d+3)(d+2)(d+1)d}{24} + 1.$

Theorem

After denoting by σ_1 and σ_2 the eigenvalues of $\boldsymbol{\sigma}$, and $\theta_1 := \theta, \theta_2 := 1-\theta$ as before, the explicit formula for the bound (3.6) is given as follows:

(i) *If*

$$\begin{aligned} &\theta_2\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| < (\theta_1\mu_2\kappa_1+\theta_2\mu_2\kappa_2+\kappa_1\kappa_2)|\sigma_1-\sigma_2| \ \& \\ &(\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)|\sigma_1+\sigma_2| \geq (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)|\sigma_1-\sigma_2|, \end{aligned} \tag{3.12}$$

then

$$\begin{aligned} \mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq (\theta_1\mathbf{A}^{-1}+\theta_2\mathbf{B}^{-1})\boldsymbol{\sigma}:\boldsymbol{\sigma}- \\ -\theta_1\theta_2\frac{[\mu_1\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2|+\kappa_1\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2|]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\mu_1\kappa_1\theta_1(\mu_2+\kappa_2)+\mu_2\kappa_2\theta_2(\kappa_1+\mu_1)]}. \end{aligned}$$

In this case the optimal microstructure for which the bound is saturated is a simple laminate with layers orthogonal to the eigenvector associated with an eigenvalue of the least absolute value, of the extremal $\boldsymbol{\eta}$ in (3.6).

(ii) *If*

$$(\kappa_2-\kappa_1)(\theta_1\mu_1+\theta_2\mu_2)|\sigma_1+\sigma_2| < (\mu_2-\mu_1)(\theta_1\kappa_1+\theta_2\kappa_2)|\sigma_1-\sigma_2|, \tag{3.13}$$

then

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq \mathbf{B}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}+\frac{\theta_1}{4}\left[\frac{(\mu_2-\mu_1)(\sigma_1-\sigma_2)^2}{\mu_2(\theta_1\mu_1+\theta_2\mu_2)}+\frac{(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)^2}{\kappa_2(\theta_1\kappa_1+\theta_2\kappa_2)}\right].$$

The optimal microstructure for which this bound is saturated is a simple laminate with layers orthogonal to $\mathbf{e}$, such that $\mathbf{e}$ is an extremal for

$$h_{\mathbf{B}}(\boldsymbol{\eta}) = \min_{\mathbf{e}\in S^1}\frac{|(\mathbf{e}\otimes\mathbf{e}):\mathbf{B}\boldsymbol{\eta}|^2}{\mathbf{B}(\mathbf{e}\otimes\mathbf{e}):(\mathbf{e}\otimes\mathbf{e})}, \tag{3.14}$$

where $h_{\mathbf{B}}$ is function of extremal $\boldsymbol{\eta}$ in (3.6).

Theorem

(iii) *If*

$$\mu_2\theta_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \geq (\theta_1\mu_2\kappa_1+\theta_2\mu_2\kappa_2+\kappa_1\kappa_2)|\sigma_1-\sigma_2|, \quad (3.15)$$

then

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq \mathbf{B}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} + \frac{\theta_1(\kappa_2-\kappa_1)(\mu_2+\kappa_2)(\sigma_1+\sigma_2)^2}{4\kappa_2[\kappa_1(\mu_2+\kappa_2)+\mu_2(\kappa_2-\kappa_1)\theta_2]}.$$

In this case, the optimal microstructure for which the bound is saturated is a rank-2 laminate with directions of lamination given by eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\sigma}$, and corresponding lamination parameters

$$m_1 = \frac{2\theta_2\mu_2(\kappa_1-\kappa_2)\sigma_2+\kappa_1(\mu_2+\kappa_2)(\sigma_1-\sigma_2)}{2\theta_2\mu_2(\kappa_1-\kappa_2)(\sigma_1+\sigma_2)},$$

$$m_2 = \frac{2\theta_2\mu_2(\kappa_1-\kappa_2)\sigma_1-\kappa_1(\mu_2+\kappa_2)(\sigma_1-\sigma_2)}{2\theta_2\mu_2(\kappa_1-\kappa_2)(\sigma_1+\sigma_2)}.$$

Cases $(i)-(iii)$ are disjoint, and the union of all $(\sigma_1,\sigma_2)\in\mathbf{R}^2$ which satisfy one of the conditions (3.12), (3.13) and (3.15), equals $\mathbf{R}^2$.

Theorem

After denoting by σ_1 and σ_2 the eigenvalues of $\boldsymbol{\sigma}$, and $\theta_1 := \theta$, $\theta_2 := 1-\theta$ as before, the explicit formula for the bound (3.9) is given as follows:

(i) *If*

$$\begin{aligned} &\theta_1\mu_1(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| < (\theta_2\mu_1\kappa_2+\theta_1\mu_1\kappa_1+\kappa_1\kappa_2)|\sigma_2-\sigma_1| \ \ \& \\ &\theta_1\kappa_1(\mu_2-\mu_1)|\sigma_2-\sigma_1| < (\theta_1\mu_1\kappa_1+\theta_2\mu_2\kappa_1+\mu_1\mu_2)|\sigma_1+\sigma_2|, \end{aligned} \tag{3.16}$$

then

$$\begin{aligned} \mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \le\ & (\theta_1\mathbf{A}^{-1}+\theta_2\mathbf{B}^{-1})\boldsymbol{\sigma}:\boldsymbol{\sigma}- \\ & -\theta_1\theta_2\frac{[\mu_1\mu_2(\kappa_1-\kappa_2)|\sigma_1+\sigma_2|+\kappa_1\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2|]^2}{4\mu_1\mu_2\kappa_1\kappa_2[\mu_1\kappa_1\theta_1(\mu_2+\kappa_2)+\mu_2\kappa_2\theta_2(\kappa_1+\mu_1)]}. \end{aligned}$$

In this case the optimal microstructure for which the bound is saturated is a simple laminate with layers orthogonal to the eigenvector associated with an eigenvalue of the largest absolute value, of the extremal $\boldsymbol{\eta}$ in (3.9).

(ii) *If*

$$\theta_1\mu_1(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \ge (\theta_2\mu_1\kappa_2+\theta_1\mu_1\kappa_1+\kappa_1\kappa_2)|\sigma_2-\sigma_1|, \tag{3.17}$$

then

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \le \mathbf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}+\frac{\theta_2(\mu_1+\kappa_1)(\kappa_1-\kappa_2)(\sigma_1+\sigma_2)^2}{4\kappa_1[\kappa_2(\mu_1+\kappa_1)+\theta_1\mu_1(\kappa_1-\kappa_2)]}.$$

The optimal microstructure for which the bound (3.9) is saturated is a rank-2 laminate with directions of lamination given by eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\boldsymbol{\sigma}$, and corresponding lamination parameters

$$\begin{aligned} m_1 &= \frac{2\theta_1\mu_1(\kappa_2-\kappa_1)\sigma_2+\kappa_2(\mu_1+\kappa_1)(\sigma_1-\sigma_2)}{2\theta_1\mu_1(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)}, \\ m_2 &= \frac{2\theta_1\mu_1(\kappa_2-\kappa_1)\sigma_1+\kappa_2(\mu_1+\kappa_1)(\sigma_2-\sigma_1)}{2\theta_1\mu_1(\kappa_2-\kappa_1)(\sigma_1+\sigma_2)}. \end{aligned}$$

Theorem

(iii) *If*

$$\theta_1\kappa_1(\mu_2-\mu_1)|\sigma_2-\sigma_1| \geq (\theta_1\mu_1\kappa_1+\theta_2\mu_2\kappa_1+\mu_1\mu_2)|\sigma_1+\sigma_2|, \quad (3.18)$$

then

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \leq \mathbf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} + \frac{\theta_2(\mu_1-\mu_2)(\mu_1+\kappa_1)(\sigma_1-\sigma_2)^2}{4\mu_1[\mu_2(\mu_1+\kappa_1)+\theta_1\kappa_1(\mu_1-\mu_2)]}.$$

In this case, the optimal microstructure for which the bound (3.9) is saturated is a rank-2 laminate with directions of lamination given with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the extremal $\boldsymbol{\eta}$ in (3.9) (which are also eigenvectors of $\boldsymbol{\sigma}$), and corresponding lamination parameters

$$m_1 = \frac{2\kappa_1\theta_1(\mu_2-\mu_1)\sigma_2-\mu_2(\mu_1+\kappa_1)(\sigma_1+\sigma_2)}{2\kappa_1\theta_1(\mu_2-\mu_1)(\sigma_2-\sigma_1)},$$

$$m_2 = \frac{-2\kappa_1\theta_1(\mu_2-\mu_1)\sigma_1+\mu_2(\mu_1+\kappa_1)(\sigma_1+\sigma_2)}{2\kappa_1\theta_1(\mu_2-\mu_1)(\sigma_2-\sigma_1)}.$$

Cases $(i)-(iii)$ are disjoint, and the union of all $(\sigma_1,\sigma_2)\in\mathbf{R}^2$ which satisfy one of the conditions (3.16), (3.17) and (3.18), equals $\mathbf{R}^2$.

Compliance minimization

For $g_1 = g_2 = fu$, it follows

$$J(\theta, \mathbf{M}) = \int_\Omega f(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} + l\int_\Omega \theta(\mathbf{x})\,d\mathbf{x}. \tag{3.19}$$

By the principle of minimal complementary energy we have

$$\int_\Omega f(\mathbf{x})u(\mathbf{x}) = \min_{\substack{\tau\in \mathrm{L}^2(\Omega;\mathrm{Sym})\\ \mathrm{div div}\tau = f \text{ in } \Omega}} \int_\Omega \mathbf{M}^{-1}\tau : \tau\,d\mathbf{x},$$

and the minimum on the right hand side is achieved by $\boldsymbol{\sigma} = \mathbf{M}\nabla\nabla u$.

Therefore, (2) can be rewritten as

$$J(\theta, \mathbf{M}) = \min_{\substack{\tau\in \mathrm{L}^2(\Omega;\mathrm{Sym})\\ \mathrm{div div}\tau = f \text{ in } \Omega}} \int_\Omega \mathbf{M}^{-1}\tau : \tau\,d\mathbf{x} + l\int_\Omega \theta(\mathbf{x})\,d\mathbf{x} \tag{3.20}$$

and

$$\min_{(\theta,\mathbf{M})\in\mathcal{A}} J(\theta, \mathbf{M}) = \min_{(\theta,\mathbf{M})\in\mathcal{A}} \min_{\substack{\tau\in \mathrm{L}^2(\Omega;\mathrm{Sym})\\ \mathrm{div div}\tau = f \text{ in } \Omega}} \int_\Omega \mathbf{M}^{-1}\tau : \tau\,d\mathbf{x} + l\int_\Omega \theta(\mathbf{x})\,d\mathbf{x}.$$

Theorem (Neccessary conditions of optimality)

If $(\theta^*, \mathbf{M}^*)$ *is a minimizer of the objective function (3.19), and if* $\boldsymbol{\sigma}^*$ *is the unique corresponding minimizer in (3.20), then* $\boldsymbol{\sigma}^* = \mathbf{M}^* \nabla\nabla u^*$*, where* u^* *is the state function for* $(\theta^*, \mathbf{M}^*)$*. Furthermore,* $\mathbf{M}^*$ *satisfies, almost everywhere in* Ω*,*

$$\mathbf{M}^{*-1}\boldsymbol{\sigma}^* : \boldsymbol{\sigma}^* = h(\theta^*, \boldsymbol{\sigma}^*) := \min_{\mathbf{M} \in G_\theta} \mathbf{M}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma}, \tag{3.21}$$

where $h(\theta^*, \boldsymbol{\sigma}^*)$ *is the lower Hashin-Shtrikman bound on the complementary energy, while* θ^* *is the unique minimizer of the convex minimization problem*

$$\min_{0 \le \theta \le 1} \left(h(\theta, \boldsymbol{\sigma}^*) + l\theta\right), \quad \textit{a.e. on } \Omega. \tag{3.22}$$

Theorem

For the objective function (3.19) we have

$$\min_{(\theta,\mathbf{M})\in\mathcal{A}} J(\theta,\mathbf{M}) = \min_{(\theta,\mathbf{M})\in\mathcal{L}^+} J(\theta,\mathbf{M}),$$

where $\mathcal{L}^+$ is the set of sequentially laminated designs defined as

$$\mathcal{L}^+ := \left\{(\theta,\mathbf{M}) \in \mathrm{L}^\infty(\Omega;[0,1]\times\mathfrak{M}_2(\alpha,\beta;\Omega)) : \mathbf{M}(\mathbf{x}) \in L^+_{\theta(\mathbf{x})},\ \textit{a.e. in}\,\Omega\right\},$$

where L^+_θ, for $\theta\in[0,1]$, is the set of all sequential laminates **M***, with core* **A** *and matrix* **B***, in proportions θ and $(1-\theta)$, respectively. If $(\theta^*,\mathbf{M}^*)$ is a minimizer of J in $\mathcal{A}$, and if $\boldsymbol{\sigma}^*$ is its associated stress tensor which minimizes (3.20), then there exists a sequential laminate $\widetilde{\mathbf{M}}$ such that $(\theta^*,\widetilde{\mathbf{M}})$ is a minimizer of J in $\mathcal{L}^+$, $\boldsymbol{\sigma}^*$ is again its associated stress tensor, and $\mathbf{M}^{*-1}\boldsymbol{\sigma}^* = \widetilde{\mathbf{M}}^{-1}\boldsymbol{\sigma}^*$.*

Algorithm

Take some initial θ^0 and $\mathbf{M}^0$. For $k \geq 0$:

1. Calculate u^k, the solution of

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^k \nabla\nabla u^k) = f \\ u^k \in \mathrm{H}_0^2(\Omega)\,, \end{cases}$$

   and define $\boldsymbol{\sigma}^k := \mathbf{M}^k \nabla\nabla u^k$.

2. For $\mathbf{x} \in \Omega$, take $\theta^{k+1}(\mathbf{x})$ as the zero of the function

$$\theta \mapsto \frac{\partial h}{\partial \theta}(\theta, \boldsymbol{\sigma}^k(\mathbf{x})) + l\,,$$

   and if a zero doesn't exist, take 0 (or 1) if the function is positive (or negative) on $[0, 1]$.

3. Let $\mathbf{M}^{k+1}(\mathbf{x})$ be the minimizer in the definition of $h(\theta^{k+1}(\mathbf{x}), \boldsymbol{\sigma}^k(\mathbf{x}))$.

Thank you for your attention!