Global Controllability for Quasilinear Non-negative Definite System of ODEs and SDEs

Darko Mitrovic

June 16, 2021

joint work with J.Djordjevic, S.Konjik, and A.Novak

The most general problem in this direction is the nonlinear system of the form

$$\dot{y}(t) = f(t, y(t), u(t)), \quad y(0) = y^0 \in \mathbb{R}^d, \tag{1}$$

and the aim is to find the vector valued function u which steers the system to a given final state $y(T) = y^T$.

The proofs of controllability of nonlinear systems essentially rely on the linear theory and Leray-Schauder-Tikhonov fixed point type theorems. Roughly speaking, we consider the Taylor expansion of f in the neighborhood of $(t, y, u) = (t, 0, 0)$:

$$f(t, y, u) = f(t, 0, 0) + f_y(t, 0, 0)y + f_u(t, 0, 0)u + g(t, y, u),$$

and then we linearize (1) by considering

$$\dot{y}(t) = f(t,0,0) + f_y(t,0,0)y + f_u(t,0,0)u + g(t,z,u),$$
$$y(0) = y^0 \in \mathbb{R}^d, \quad y(T) = y^T \in \mathbb{R}^d, \tag{2}$$

for a fixed function $z : [0, T] \to \mathbb{R}^d$ of appropriate regularity.
One proves or assumes existence of the control u such that (2) holds and thus one obtains the mapping

$$\mathcal{T}(z) = y$$

which adjoins the state y to the previously fixed function z. Clearly, if we prove existence of a fixed point for the mapping $\mathcal{T}$ we will prove existence of control to (1).

Here, we shall consider an ODE system with quasilinear non-negative definite symmetric right-hand side

$$\dot{y}(t) = -A(t, y(t))y(t) + B(t, y(t))u(t), \quad y(0) = y^0, \quad y(T) = y^T, \quad (3)$$

where $(t, y) \mapsto A(t, y) \in \mathbb{R}^{d\times d}$ and $(t, y) \mapsto B(t, y) \in \mathbb{R}^{d\times n}$ are matrix-valued functions of appropriate regularity.

We shall imply:

(i) $A : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a smooth non-negative definite symmetric matrix;

(ii) $B : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is a matrix-valued L^∞-uniformly bounded mapping;

(iii) For every $v = (v_1, \dots, v_d)$, $v_j \in L^2(\Omega; C([0, T]))$, $j = 1, \dots, d$, the Gramian (note the difference in the notation of the final time T and the transpose T below)

$$G_c(0, T) = \int_0^T e^{\int_0^t -A(t,v)dt'} B(t, v) B(t, v)^{\mathrm{T}} \left(e^{\int_0^t (-A(t,v))dt'} \right)^{\mathrm{T}} dt \tag{4}$$

is invertible.

Proposition

Let $A \in \mathbb{R}^{d\times d}$ *and* $B \in \mathbb{R}^{d\times m}$ *be constant matrices. If the Kalman rank condition*

$$\mathrm{rank}\big[B\big|AB\big|\dots\big|A^{d-1}B\big] = d \tag{5}$$

holds, then the Gramian

$$G_c(0,T) = \int_0^T e^{\int_0^t(-|v|^m A)dt'} BB^{\mathrm{T}} \left(e^{\int_0^t(-|v|^m A)dt'}\right)^{\mathrm{T}} dt \tag{6}$$

corresponding to

$$y'(t) = -A|v(t)|^m y(t) + Bu(t), \tag{7}$$

where $A \in \mathbb{R}^{d\times d}$ *and* $B \in \mathbb{R}^{d\times n}$ *are constant matrices, is invertible for every* $v \in C([0,T])$ *(in other words, (iii) holds).*

Proof sketch

We consider the perturbation:

$$y_\varepsilon'(t) = (-A|v(t)|^m - \varepsilon A)y_\varepsilon(t) + Bu_\varepsilon(t), \quad \varepsilon > 0. \tag{8}$$

and prove equivalence of controlability of the latter system with the Kalman condition using the standard Kalman approach.

By letting $\varepsilon \to 0$ we see that limit functions y and u satisfy

$$y'(t) = -A|y(t)|^m y(t) + Bu(t), \qquad y(0) = y^0,$$

with appropriate initial and end conditions implying controllability of the system.

Background from functional analysis

The linear mapping $K : X \to Y$, where X and Y are the Banach spaces is surjective if and only if the dual mapping $K^* : Y^* \to X^*$ is coercive i.e.

$$|K^* y^*|_{X^*} \geq \delta |y^*|_{Y^*}.$$

If X and Y are reflexive Banach spaces, then the latter is equivalent with the existence of global minimum for every $y \in Y$ of the functional

$$\mathcal{J}(y, y^*) = \frac{1}{2} |K^* y^*|^2_{X^*} + \langle y, y^* \rangle.$$

We note that the Eurler-Lagrange will provide and explicit expression for y^*.

Leray-Schauder-Tikhonov fixed point thm

Let T be a continuous and compact mapping of a Banach space X into itself, such that the set

$$\{x \in T : x = \lambda Tx\}$$

is bounded.
Then, T has a fixed point.

We are concerned with a control problem for the nonlinear system of ODEs of the following form

$$\frac{dy}{dt}(t) = -A(t, y(t))y(t) + B(t, y(t))u(t), \quad t \in [0, T], \tag{9}$$

$$y(0) = y^0, \tag{10}$$

where $A : [0, T] \times \mathbb{R} \to \mathbb{R}^{d\times d}$ is a smooth symmetric positive semi-definite matrix-valued mapping, $B : [0, T] \times \mathbb{R} \to \mathbb{R}^{d\times n}$ is a smooth matrix-valued mapping uniformly bounded with respect to L^∞-norm, and $y^0 \in \mathbb{R}^d$ is given initial state. The problem consists of determining a function $u \in L^2([0, T]; \mathbb{R}^n)$ such that for the given final state $y^T \in \mathbb{R}^d$ and the final time T

$$y(T) = y^T. \tag{11}$$

First, we shall linearize equation (9) as follows

$$\frac{dy}{dt}(t) = -A(t, v(t))y(t) + B(t, v(t))u(t), \quad t \in [0, T], \tag{12}$$

where $v \in C([0, T])$ is fixed. For such a fixed v, we look for the control u steering the system from the initial state y^0 to the prescribed final state y^T.

We denote

$$\mathcal{T}(v) = y.$$

In our case, we will aim to prove that the solution y and its derivative y' are bounded independently of v and u which will provide the compactness of the mapping $\mathcal{T}$ i.e. it will provide existence of the fixed point y being the solution to the non-linearized problem.

The adjoint system associated to the linearized control problem is given by

$$-\frac{d\varphi}{dt}(t) = - A^*(t, v(t))\varphi(t), \quad t \in [0, T], \tag{13}$$

$$\varphi(T) = \varphi^T, \tag{14}$$

where $\varphi^T \in \mathbb{R}^d$ minimizes the convex functional

$$J(\varphi^T) = \int_0^T |B^*(t, v(t))\varphi(t)|^2 dt + \langle y^0, \varphi(0)\rangle - \langle y^T, \varphi^T\rangle. \tag{15}$$

In the language of the functional analysis mentioned above, we take

$$Ku = y^T, \quad K^*(\varphi^T) = B^*\varphi.$$

Thus, we have:

A solution to problem (12) is then given by $u(t) = B^*\varphi(t)$ and we have

$$\begin{aligned} |u(t)| &= \left|B^*(t,v(t))\varphi(t)\right| = \left|B^*(t,v(t))e^{\int_t^T -A(s,v(s))\,ds}\varphi^T\right| \\ &\leq C|\varphi^T|, \end{aligned} \tag{16}$$

where φ is a solution to the corresponding adjoint problem and C is independent of T.

We have thus bound on u and using the non-positive definiteness of the matrix of the system, we are able to obtain bound of y independent of u and v.
We need bound of y' to get the wanted compactness of $\mathcal{T}$.

To this end, introduce a change of variables $\tau = K_1 t$ into (12), for some parameter $K_1 > 0$. Denote by

$$\tilde{v}(\tau) = v\Big(\frac{\tau}{K_1}\Big), \quad \tilde{u}(\tau) = u\Big(\frac{\tau}{K_1}\Big), \quad \tilde{y}(\tau) = y\Big(\frac{\tau}{K_1}\Big),$$
$$\tilde{A}(\tau, \tilde{v}(\tau)) = A(\frac{\tau}{K_1}, \tilde{v}(\tau)), \quad \tilde{B}(\tau, \tilde{v}(\tau)) = B(\frac{\tau}{K_1}, \tilde{v}(\tau)).$$

Since an arbitrary $\tilde{v} \in \mathcal{C}([0, K_1 T])$ is bounded, it follows from (16) that $\tilde{u}$ is also bounded. Thus for any $\tilde{v}$ one can choose a constant $K_1 \geq 1$ such that

$$\left|\frac{\tilde{A}(\cdot, \tilde{v}(\cdot))}{K_1}\right| \leq 1 \quad \text{and} \quad \left|\tilde{B}(\cdot, \tilde{v}(\cdot))\frac{\tilde{u}}{K_1}\right| \leq 1.$$

$$\frac{d\tilde{y}}{d\tau} = -\frac{\tilde{A}(\tau, \tilde{v}(\tau))}{K_1}\tilde{y}(\tau) + \tilde{B}(\tau, \tilde{v}(\tau))\frac{\tilde{u}(\tau)}{K_1}, \quad \tau \in [0, K_1 T], \tag{17}$$

$$\tilde{y}(0) = y^0, \quad \tilde{y}(K_1 T) = y^T. \tag{18}$$

Next, introduce an operator $\mathcal{T} : \mathcal{C}([0, T]) \to \mathcal{C}([0, T])$, which maps $\tilde{v}$ to a solution $\tilde{y}$ of (17), (18) (more precisely to the first component of the solution $(\tilde{y}, \tilde{u})$) i.e.,

$$\mathcal{T}(\tilde{v}) = \tilde{y}.$$

Furthermore, the solution of (17), (18) is given by

$$\tilde{y}(\tau) = \exp\Big(-\int_0^\tau \frac{\tilde{A}(s,\tilde{v}(s))}{K_1}\,ds\Big)\Big[\int_0^\tau \exp\Big(\int_0^s \frac{\tilde{A}(\theta,\tilde{v}(\theta))}{K_1}\,d\theta\Big)\times \\ \times \tilde{B}(s,\tilde{v}(s))\frac{\tilde{u}(s)}{K_1}\,ds + y^0\Big]. \tag{19}$$

one can estimate $\tilde{y}$ as follows

$$|\tilde{y}(\tau)| \leq \Big| \exp\Big(-\int_0^\tau \frac{\tilde{A}(s,\tilde{v}(s))}{K_1}\,ds\Big)\Big| \Big[\int_0^\tau \Big| \exp\Big(\int_0^s \frac{\tilde{A}(\theta,\tilde{v}(\theta))}{K_1}\,d\theta\Big)\mathbf{1}\Big|\,ds + |y^0|\Big] \leq \tau + |y^0|, \tag{20}$$

where $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^d$. This estimate together with (17) implies

$$|\partial_\tau \tilde{y}| \leq \tau + |y^0| + 1. \tag{21}$$

Therefore, by the Arzela-Ascolli theorem, the operator $\mathcal{T} : C([0, T]) \to C([0, T])$ is compact. Moreover, the set

$$\{x \in \mathbb{C}([0, T]) : \, x = \lambda T x, \lambda \in [0, 1]\}$$

is bounded due to (20).

The latter implies that the conditions of the Leray-Schauder fixed point theorem are satisfied for the operator $\mathcal{T}$ which in turn implies existence of a fixed point $\tilde{y}$ of $\mathcal{T}$ that satisfies

$$\frac{d\tilde{y}}{d\tau} = -\frac{\tilde{A}(\tau, \tilde{y}(\tau))}{K_1}\tilde{y}(\tau) + \tilde{B}(\tau, \tilde{y}(\tau))\frac{\tilde{u}(\tau)}{K_1}, \quad \tau \in [0, T].$$

By reintroducing the change of variables $t = \tau/K_1$ we obtain the solution $y_1(t) = \tilde{y}(t/K_1)$ to (9) on the interval $[0, T/K_1]$.
We now repeat the whole procedure for the problem (9) with the initial data given at T/K_1, i.e., $y(T/K_1) = y_1(T/K_1)$, and the same final state $y(T) = y^T$. We thus obtain the function y_2 representing the solution to (9) on the interval $[T/K_1, T/K_1 + T/K_2]$ etc...

A numerical example

We shall consider population dynamics with species intending to avoid crowding. It is interesting to inspect how to control the populations by adding new individuals or in some other way by improving the living conditions (in the frame of the given nonlinear model, of course). We assume that we have two species whose population quantities are denoted by y_1 and y_2 and which have tendency to avoid crowding. Mathematical model of the phenomenon is given by

$$\begin{aligned} dy_1 &= (|y_1 + y_2|(-2y_1 + 2y_2) + u)dt + Z_1 dW_t \\ dy_2 &= |y_1 + y_2|(y_1 - y_2)dt + Z_2 dW_t \end{aligned} \tag{22}$$

and we aim to maintain the population by randomly introducing new individuals u_1 and u_2 of the corresponding species into the system.

we start with the populations

$$y_1(0,\omega) = y_2(0,\omega) = 1,$$

and we want to have the same population at a final time $T = 0.5$

$$E(y_1(0.5,\cdot)) = E(y_2(0.5,\cdot)) = 2. \tag{23}$$

By direct substitution, we know that for the problem

$$dy = (A(t)y + B(t)u)dt + ZdW_t, \quad E(y(0,\cdot)) = y_0, \quad E(y(0.5,\cdot)) = y_1,$$

the control function is given by

$$u(t,\omega) = -B^{\mathrm{T}}\Phi^{\mathrm{T}}(0,t)W_c(0,1)^{-1}[y_0 - \Phi(0,1)y_1] \tag{24}$$

where

$$W_c(0,t) = \int_0^t \Phi(0,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(0,\tau)d\tau \tag{25}$$

and Φ solves the system

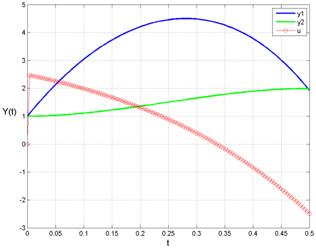
$$\Phi' = A(t)\Phi, \quad \Phi(0,0) = I. \tag{26}$$

We use the following recursive procedure

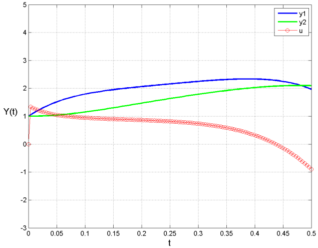
$$
\begin{aligned}
dy_1^n &= (|y_1^{n-1} + y_2^{n-1}|(-2y_1^n + 2y_2^n) + u^n)dt + Z_1^n dW_t \\
dy_2^n &= |y_1^{n-1} + y_2^{n-1}|(y_1^n - y_2^n)dt + Z_2^n dW_t
\end{aligned}
$$

where u^n is given by the corresponding variant of (24).

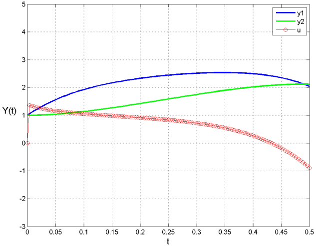
Simulation results



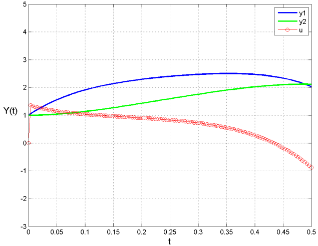
(a) Initial iteration



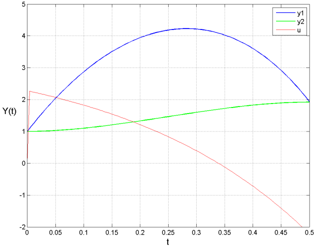
(b) After two iterations.



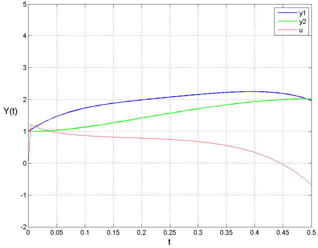
(c) After three iterations.



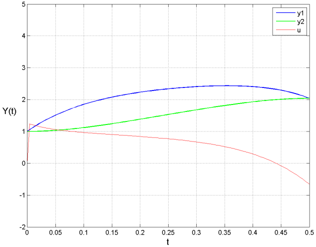
(d) After four iterations.



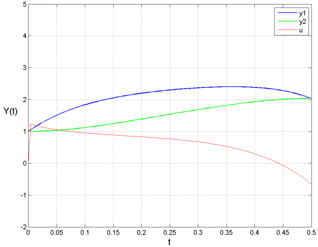
(a) Initial iteration



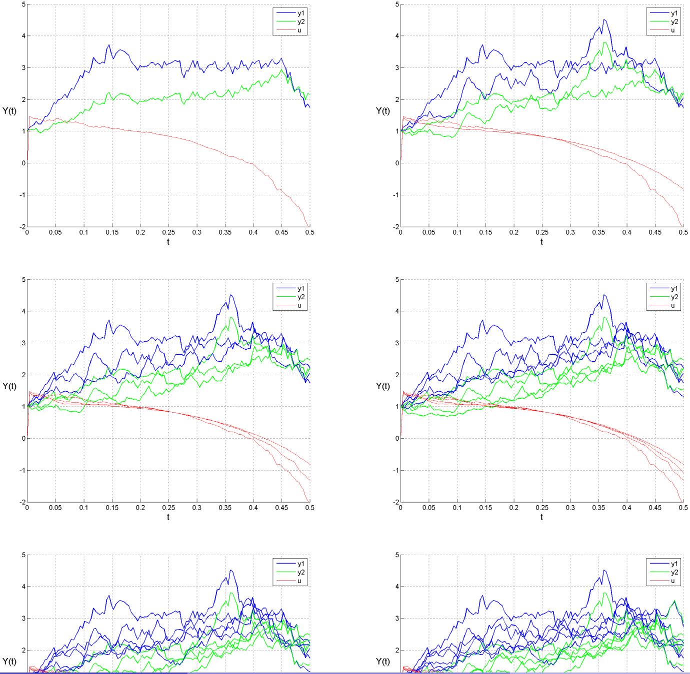
(b) After two iterations.



(c) After three iterations.



(d) After four iterations.



y1
y2
u
Y(t)
t