Robust control and Stackelberg strategy for a fourth–order parabolic equation

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Hierarchic control

Hierarchic control

- Concept from game theory (Gabriel Cramer 1728, Daniel Bernoulli 1738).
- What is now known as Nash equilibria is due to Cournot (1838).
- Historical papers due to J. Von Neumann and O. Morgenstern (1943) and J. Nash (1950).

Stackelberg strategy?

- One of the players (the leader) has some advantage that allows her to commit to a strategy.
- The other player (the follower) then chooses his best response to this.
- The leader (first player) does a movement. The follower (second player) reacts trying to win or optimize the response to the leader movement.

We consider the heat equation

$$\begin{cases} u_t - \Delta u = \overbrace{h1_\omega}^{leader\ control} + \overbrace{v1_{\mathcal{O}}}^{follower\ control} & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \qquad \text{(PS)}$$

Objectives:

1. *Optimal control*: $u \approx u_d$ in $\mathcal{O}_d \times (0,T)$

$$\min_{v \in L^2(Q)} \frac{1}{2} \iint_{\mathcal{O}_d \times (0,T)} |u - u_d|^2 dxdt + \frac{\beta}{2} \iint_{\mathcal{O} \times (0,T)} |v|^2 dxdt, \quad \beta > 0.$$

2. *Null controllability*: find h such that $u(T) = 0$.

Step 1. Fix h and obtain

$$\min_{v\in L^2(Q)} \quad \frac{1}{2}\iint_{\mathcal{O}_d\times(0,T)} |u-u_d|^2 dxdd + \frac{\beta}{2}\iint_{\mathcal{O}\times(0,T)} |v|^2 dxdt.$$

The functional is continuous, strictly convex and coercive so there is a unique minimizer characterized by

$$\boxed{v = -\frac{1}{\beta} p\chi_{\mathcal{O}}}$$

$$\begin{cases} -p_t - \Delta p = (u-u_d)\chi_{\mathcal{O}_d} \quad \text{in } \Omega\times(0,T), \\ p(x,T) = 0 \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega\times(0,T). \end{cases} \qquad \text{(AS)}$$

Step 2. Consider the coupled system:

$$\begin{cases} u_t - \Delta u = h 1_\omega - \frac{1}{\beta} p \chi_{\mathcal{O}} & \text{in } \Omega \times (0,T), \\ -p_t - \Delta p = (u - u_d)\chi_{\mathcal{O}_d} & \text{in } \Omega \times (0,T), \\ u = p = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \;\; p(x,T) = 0, & \text{in } \Omega. \end{cases} \tag{PS-AS}$$

How to solve the problem?

Step 2. Consider the coupled system:

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☞ [Lions94]: system (PS-AS) is approximately controllable, i.e.,

$$\|u(T)\|_{L^2(\Omega)} \leq \varepsilon.$$

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☞ [Araruna et al. (2015)]: when $\omega \cap \mathcal{O}_d \neq \emptyset$ and

$$\boxed{\iint_{\Omega \times (0,T)} \rho^2 |u_d|^2 dxdt < +\infty \quad \text{for } \rho \to \infty \text{ as } t \to T,}$$

then (PS-AS) is NULL CONTROLLABLE.

Works related with hierarchic control

- Heat and wave equations: Lions, 1994.
- Ocean circulation models: Díaz-Lions 1997, Díaz (2002).
- Stokes system: Guillen-González et al., approximate control, 2013.
- Moving Domains (wave equation): IP de Jesus, 2014, 2015.
- Moving domains (Parabolic equations) Approximate control: Límaco, J.; Clark, H. R.; Medeiros, L. A. , 2009.
- Linear and semilinear parabolic equations: Araruna, Fernández-Cara, Santos, 2015 –Control to trajectories.
- Micropolar fluids (linear case): Araruna, F. D.; de Menezes, S. D. B.; Rojas-Medar, M. A. 2014 – approximate controllability, control in both equations.
- Coupled parabolic equations: Hernández–Santamaría; DeT, Pozniak (2016).

Robust Control

Robust control

– A system is said to be robust when:

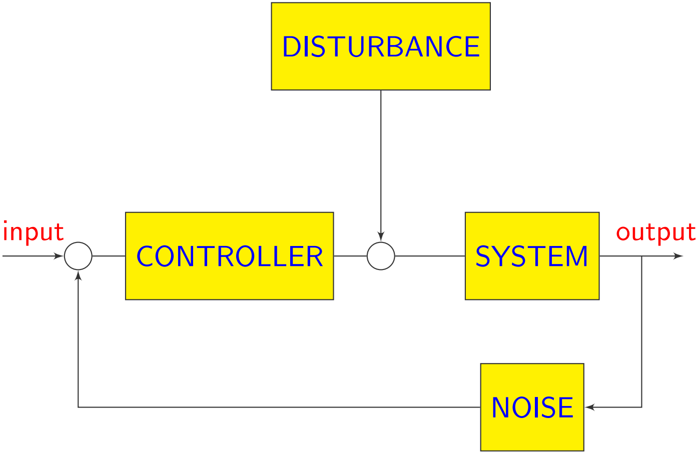
- It is hardy, durable and resilient.
- It has low sensitivities in the system passband.
- It is stable over the range of parameter variations.
- The performance continues to meet the specifications in the present of set of changes in the system parameters

– Robustness is the sensitivity to the effects that are not considered in the analysis and design: for example

disturbance signals

noise measurements

Robust control



Robust control

- Two important problems that are often encountered: a disturbance signal is added to the control input to the system. That can account for wind gusts in airplanes, changes in ambient temperature in ovens, etc., and noise that is added to the sensor output.
- A differential game between an engineer seeking the best control which stabilizes the perturbation with limited control effort and simultaneously, nature seeking maximally malevolent disturbance which destabilizes the perturbation with limited disturbance magnitude.
- Optimal control problem: Find a saddle point. Minimize with respect to a control, maximize with respect to the disturbance.

Stackelberg strategy for robust control

General framework

$Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$, $\mathcal{A}, \mathcal{N}$ appropriate operators.
We consider

$$\begin{cases} y_t - \mathcal{A}y + \mathcal{N}y = h\chi_\omega + v\chi_\mathcal{O} + \psi & \text{in } Q, \\ +BC & \text{on } \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{in } \Omega. \end{cases} \tag{1}$$

h – leader control v – follower control ψ – perturbation.

Remarks:

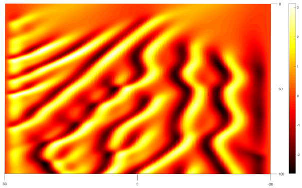
1. $h \equiv 0$ or $v \equiv 0 \Rightarrow$ robust control problem.
2. $\psi \equiv 0 \Rightarrow$ Stackelberg strategy.
3. $h, v, \psi \neq 0 \Rightarrow$ Robust Stackelberg controllability.

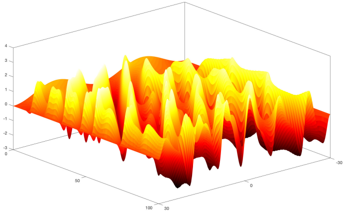
Our model

Domain: $Q = (0,1) \times (0,T)$.

$$\overbrace{\begin{aligned} & y_t + y_{xxxx} + y_{xx} + yy_x = f, \\ & +BC, \\ & y(\cdot,0) = y_0(\cdot). \end{aligned}}^{\text{Kuramoto–Sivashinsky equation}}$$

- Phase turbulence in reaction diffusion systems; diffusive instabilities in a laminar flame.
- y_{xxxx}: dissipative term ; provides damping at small scales.
- y_{xx}: an instability at large scales.
- yy_x: stabilizes by transferring energy between large and small scales.





Our problem: Robust Stackelberg controllability

$Q := (0,1) \times (0,T)$, $\Sigma := \{0,1\} \times (0,T)$, $\omega, \mathcal{O} \subset (0,1)$.

We consider the Kuramoto–Sivashinsky equation:

$$\begin{cases} y_t - y_{xxxx} + y_{xx} + yy_x = h\chi_\omega + v\chi_{\mathcal{O}} + \psi, & \text{in Q}, \\ y(0,t) = y(1,t) = y_x(0,t) = y_x(1,t) = 0 & \text{on } \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{in } \Omega. \end{cases} \tag{2}$$

h – leader control v – follower control ψ – perturbation.

- Step 1: EXISTENCE, UNIQUENESS AND CHARACTERIZATION.
  Fix $h \in L^2(0,T;L^2(\omega))$. Find the saddle point for

$$J_r(v,\psi) = \frac{1}{2}\|y - y_d\|^2_{L^2(\mathcal{O}_d \times (0,T))} + \frac{\ell^2}{2}\|v\chi_{\mathcal{O}}\|^2_{L^2(Q)} - \frac{\gamma^2}{2}\|\psi\|^2_{L^2(Q)}.$$

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Idea of the proof

- Step 1: EXISTENCE, UNIQUENESS AND CHARACTERIZATION.
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Theorem (Convex analysis)

Let J be a functional defined on $X \times Y$, where X and Y are convex, closed, non–empty, unbounded sets. If

1. $\forall v \in X, \quad \psi \longmapsto J(v,\psi)$ *is concave and upper semicontinuous.*
2. $\forall \psi \in Y, \quad v \longmapsto J(v,\psi)$ *is convex and lower semicontinuous.*
3. $\exists v_0 \in X$ *such that* $\lim_{\|\psi\|_Y\to\infty} J(v_0,\psi) = -\infty$
4. $\exists \psi_0 \in Y$ *such that* $\lim_{\|v\|_X\to\infty} J(v,\psi_0) = +\infty$

Then J possesses at least one saddle point $(\bar{v},\bar{\psi})$ and

$$J(\bar{v},\bar{\psi}) = \min_{v\in X}\sup_{\psi\in Y} J(v,\psi) = \max_{\psi\in Y}\inf_{v\in X} J(v,\psi).$$

1) γ, ℓ large enough and small data $\Rightarrow \forall h \in L^2(0,T;L^2(\omega))$, $\exists!$ saddle point $(\overline{v}, \overline{\psi})$ characterized by

$$\bar{v} = -\frac{1}{\ell^2} z \chi_{\mathcal{O}}, \qquad \bar{\psi} = \frac{1}{\gamma^2} z,$$

$$\begin{cases} -z_t + z_{xxxx} + z_{xx} - y z_x = (y - y_d) 1_{\mathcal{O}_d} & \text{in } Q, \\ z(0,t) = z(1,t) = z_x(0,t) = z_x(1,t) = 0 & \text{on } \Sigma, \\ z(\cdot, T) = 0 & \text{in } (0,1). \end{cases}$$

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- Step 2: Show the local null controllability for

$$\begin{cases} y_t + y_{xxxx} + y_{xx} + yy_x = h1_\omega + (-\ell^{-2}1_{\mathcal{O}} + \gamma^{-2})z & \text{in } Q, \\ -z_t + z_{xxxx} + z_{xx} - yz_x = (y - y_d)1_{\mathcal{O}_d} & \text{in } Q, \\ y(0,t) = y(1,t) = z(0,t) = z(1,t) = 0 & \text{on } \Sigma, \\ y_x(0,t) = y_x(1,t) = z_x(0,t) = z_x(1,t) = 0 & \text{on } \Sigma, \\ y(\cdot,0) = y_0(\cdot), \;\; z(\cdot,T) = 0 & \text{in } (0,1). \end{cases} \tag{3}$$

- Step 2: SHOW THE LOCAL NULL CONTROLLABILITY FOR

$$\begin{cases} y_t + y_{xxxx} + y_{xx} + yy_x = h1_\omega + (-\ell^{-2}1_{\mathcal{O}} + \gamma^{-2})z & \text{in } Q, \\ -z_t + z_{xxxx} + z_{xx} - yz_x = (y - y_d)1_{\mathcal{O}_d} & \text{in } Q, \\ y(0,t) = y(1,t) = z(0,t) = z(1,t) = 0 & \text{on } \Sigma, \\ y_x(0,t) = y_x(1,t) = z_x(0,t) = z_x(1,t) = 0 & \text{on } \Sigma, \\ y(\cdot,0) = y_0(\cdot),\ z(\cdot,T) = 0 & \text{in } (0,1). \end{cases} \tag{3}$$

$$\downarrow$$

Linear case: Observability (...Carleman estimates)

$$\begin{cases} -\varphi_t + \varphi_{xxxx} + \varphi_{xx} = g_1 + \theta 1_{\mathcal{O}_d} & \text{in } Q, \\ \theta_t + \theta_{xxxx} + \theta_{xx} = g_2 - \ell^{-2}\varphi 1_{\mathcal{O}} + \gamma^{-2}\varphi & \text{in } Q, \\ \varphi(0,t) = \varphi(1,t) = \theta(0,t) = \theta(1,t) = 0 & \text{on } \Sigma, \\ \varphi_x(0,t) = \varphi_x(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0 & \text{on } \Sigma, \\ \varphi(\cdot,T) = \varphi_T(\cdot),\ \theta(\cdot,0) = 0 & \text{in } (0,1). \end{cases} \tag{4}$$

Linear case: Observability (... Carleman estimates)

$$\begin{cases} -\varphi_t + \varphi_{xxxx} + \varphi_{xx} = g_1 + \theta 1_{\mathcal{O}_d} & \text{in } Q, \\ \theta_t + \theta_{xxxx} + \theta_{xx} = g_2 - \ell^{-2}\varphi 1_{\mathcal{O}} + \gamma^{-2}\varphi & \text{in } Q, \\ \varphi(0,t) = \varphi(1,t) = \theta(0,t) = \theta(1,t) = 0 & \text{on } \Sigma, \\ \varphi_x(0,t) = \varphi_x(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0 & \text{on } \Sigma, \\ \varphi(\cdot,T) = \varphi_T(\cdot),\ \theta(\cdot,0) = 0 & \text{in } (0,1). \end{cases} \tag{5}$$

Observability inequality

$$\|\varphi(\cdot,0)\|^2_{L^2(Q)^N} + \iint_Q \rho_1(t)|\varphi|^2 dxdt + \iint_Q \rho_2(t)|\theta|^2 dxdt$$

$$\leq C\left(\iint_Q \rho_3(t)(|g_1|^2 + |g_2|^2)dxdt + \iint_{\omega\times(0,T)} \rho_4(t)|\varphi|^2 dxdt\right).$$

$\rho_j(t)$: Carleman weights, $j = 1, \ldots, 4$.

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Theorem (L. Breton., C.M, 2021)

Assume that $\omega \cap \mathcal{O}_d \neq \emptyset$. $\forall T > 0$, $\omega \cap \mathcal{O} = \emptyset$, γ, ℓ are large enough and $\delta > 0$ small. $\exists\, \rho, \rho \to +\infty$, $t \to T$ such that

$$\iint_{\mathcal{O}_d \times (0,T)} \rho^2 |y_d|^2 < +\infty \quad \text{and} \quad \|y_0\|_{L^2(0,1)} \leq \delta.$$

Then

$$\exists \text{ null control } h \;\&\; \exists! \text{ saddle point } (\bar{v}, \bar{\psi}).$$

- V.Hernández–Santamaría, L de Teresa, *Robust Stackelberg controllability for linear and semilinear heat equations*, Evol. Equ. Control Theory, 7(2): 247-273, 2018.
- C. Montoya, L de Teresa, *Robust Stackelberg controllability for the Navier–Stokes equations.* NoDEA Nonlinear Differential Equations Appli., 25(5): Art. 46, 33, 2018.
- L. Breton., C. Montoya, *Robust Stackelberg controllability for the Kuramoto–Sivashinsky Equation*. Under review. https://arxiv.org/abs/2005.13060

Numerical experiments...robust control

Numerical scheme for the Kuramoto–Sivashinsky eq:
θ–scheme/Adams–Bashforth (time); $\mathbb{P}_1$–FE (space):

$$\frac{u^{n+1}-u^n}{\Delta t} + \theta\mathcal{A}(w^{n+1}) + (1-\theta)\mathcal{A}(w^n) - \frac{3}{2}\mathcal{N}(u^n) + \frac{1}{2}\mathcal{N}(u^{n-1}) = f^{n+1},$$
$$w^{n+1} - u_{xx}^{n+1} = 0,$$

$$V_h = \{u \in C([-L,L]) : u|_{[x_j,x_{j+1}]} \in \mathbb{P}_1 \text{ for all } 0 \leq j \leq N\}$$

and its subspace

$$V_{0h} = \{u \in V_h : u(-L) = u(L) = 0\}.$$

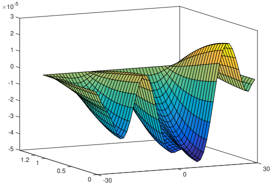
Errors between exact and approximate solutions

Δt	N	L^∞ − error	L^2 − error
1e−01	200	$1.32e-02$	$5.54e-06$
1e−02		$1.13e-03$	$3.46e-08$
1e−03		$8.79e-05$	$2.71e-10$
1e−04		$5.55e-05$	$5.66e-11$
1e−05		$5.46e-05$	$6.58e-11$
1e−06		$5.45e-05$	$6.70e-11$
1e−06	25	$3.31e-03$	$1.86e-06$
	50	$8.83e-04$	$6.58e-08$
	100	$2.19e-04$	$2.13e-09$

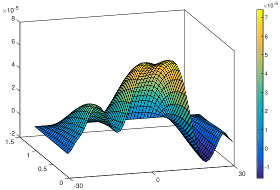
Example...robust control

Disturbance signals ψ (left) and control functions v (right) on the spatial domain $(-30, 30)$. $T = 1s$, $N = 50, \Delta t = 2 \times 10^{-2}$. $\ell = 40, \gamma = 40$ (top); $\ell = 40, \gamma = 400$, $\mathcal{O} = (-10, 10)$ (bottom).

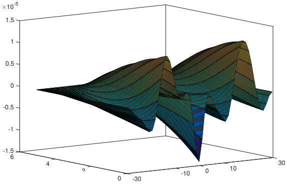
Disturbance function ψ_1



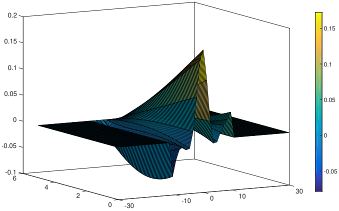
Control function v_1



Disturbance function ψ_2



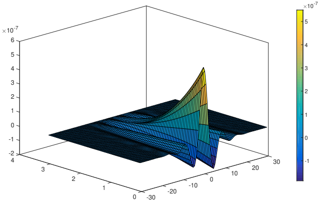
Control function v_2



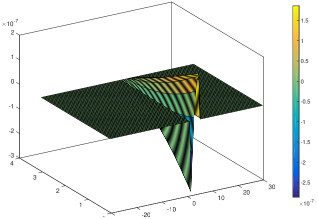
Example...Robust Stackelberg controllability

Robust Stackelberg controllability: $T = 3s, N = 100, \Delta t = 2 \times 10^{-2}, \ell = \gamma = 40$.
Domains $\omega = (-3, 1)$ and $\mathcal{O} = (2, 5)$, initial datum $u_0(x) = 10^{-3} \exp(-x^2)$.

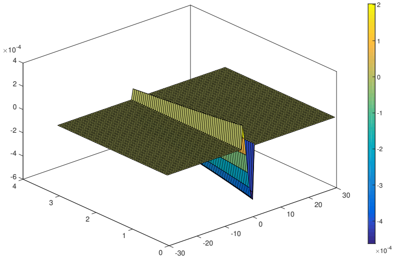
Disturbance function ψ



Follower function v



Lider function h



Open problems

- Does it occurs the null controllability when the leader control h is located on the boundary?
- Is it possible to consider a Nash–Stackelberg strategy instead of Stackelberg strategy?
- Is it possible to study this scheme to other models (KdV, micropolar fluids, Boussinesq system, . . .)?
- Efficient numerical methods for solving robust–Stackelberg controllability problems.

Thank you