Damping optimization in mechanical systems using parametric model reduction

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We consider vibrational system

$$M\ddot{q}(t) + \overbrace{(C_{int} + B_2 G B_2^T)}^{C(g)=\text{damping part}}\dot{q}(t) + Kq(t) = E_2 w(t),$$
$$y(t) = H_1 q(t).$$

- $M, K \in \mathbb{R}^{n\times n}$ mass and stiffness, the symmetric and positive definite
- $q \in \mathbb{R}^n$ state vector and y is output vector determined by $H_1 \in \mathbb{R}^{\ell\times n}$,
- $E_2 \in \mathbb{R}^{n\times m}$ determines primary excitation matrix and vector $w \in \mathbb{R}^m$ corresponds to primary excitation input.
- $C_{int} \in \mathbb{R}^{n\times n}$ internal damping e.g. $C_{int} = \alpha_c C_{crit}$, where

$$C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2},$$

- $G = \operatorname{diag}(g_1, g_2, \ldots, g_k)$, $g_i \geq 0$ damping coefficients.

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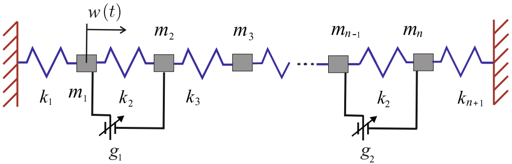
Example: *n-mass oscillator or oscillator ladder*



$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n), \quad C(g) = \alpha_c C_{crit} + B_2 G B_2^T,$$

$$B_2 G B_2^T = g_1(e_i - e_{i+1})(e_i - e_{i+1})^T + g_2(e_j - e_{j+1})(e_j - e_{j+1})^T.$$

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix}.$$

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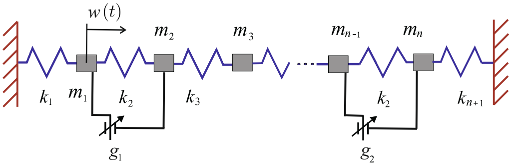

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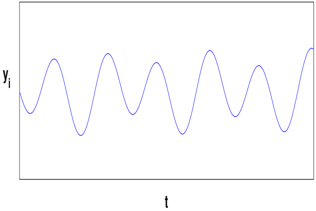

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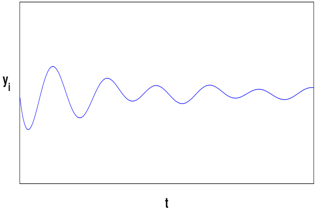

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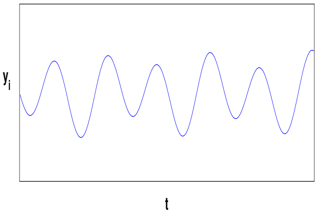

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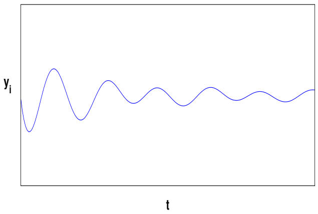

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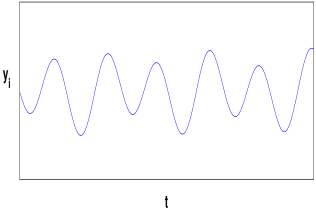

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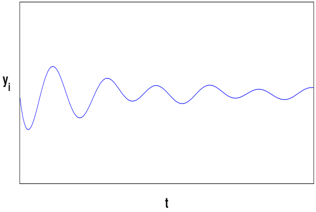

Linearization

With the substitutions $x_1(t) := q(t)$, $x_2(t) := \dot{q}(t)$ and $x := \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ we obtain a first-order representation of the closed-loop system

$$\overbrace{\begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix}}^{\mathcal{E}} \dot{x}(t) = \overbrace{\begin{bmatrix} 0 & I_n \\ -K & -C(g) \end{bmatrix}}^{A} x(t) + \overbrace{\begin{bmatrix} 0 \\ E_2 \end{bmatrix}}^{B} w(t),$$
$$y(t) = \underbrace{\begin{bmatrix} H_1 & 0 \end{bmatrix}}_{C} x(t).$$

Using the Laplace transform we obtain the closed-loop transfer function

$$\begin{aligned} \mathbf{H}(g,s) &= H_1 \left(s^2 M + sC(g) + K\right)^{-1} E_2 \\ &= \begin{bmatrix} H_1 & 0 \end{bmatrix} \left(s \begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix} - \begin{bmatrix} 0 & I_n \\ -K & -C(g) \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ E_2 \end{bmatrix} \end{aligned}$$

$\mathcal{H}_2$ norm of a system

Define the space

$$\mathcal{H}_2^{\ell\times m} := \left\{ \mathbf{H} : \mathbb{C}^+ \to \mathbb{C}^{\ell\times m} \;\middle|\; \mathbf{H} \text{ is analytic in } \mathbb{C}^+ \text{ and } \int_{-\infty}^{+\infty} \operatorname{tr}\left(\mathbf{H}(\mathrm{i}\omega)^*\mathbf{H}(\mathrm{i}\omega)\right) d\omega < \infty \right\},$$

$$\|\mathbf{H}(g,\cdot)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \operatorname{tr}\left(\mathbf{H}(g,\mathrm{i}\omega)^*\mathbf{H}(g,\mathrm{i}\omega)\right) d\omega \right)^{\frac{1}{2}}.$$

It can be expressed via the solution of a Lyapunov equation, i.e.

$$\|\mathbf{H}(g,\cdot)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \operatorname{tr} B^T \mathcal{P} B \right)^{\frac{1}{2}}, \quad \text{where} \quad A^T\mathcal{P} + \mathcal{P}A = -C^T C$$

[T./Beattie/Gugercin18, Benner/Kurschner/T./Truhar16]

$\mathcal{H}_\infty$ norm of a system [1]

Define the space

$$\mathcal{H}_\infty^{\ell\times m} := \left\{ \mathbf{H} : \mathbb{C}^+ \to \mathbb{C}^{\ell\times m} \;\middle|\; \mathbf{H} \text{ is analytic in } \mathbb{C}^+ \text{ and } \sup_{s\in\mathbb{C}^+} \|\mathbf{H}(s)\|_2 < \infty \right\}$$

$$\|\mathbf{H}(g,\cdot)\|_{\mathcal{H}_\infty} := \sup_{\lambda\in\mathbb{C}^+} \|\mathbf{H}(g,\lambda)\|_2 = \sup_{\omega\in\mathbb{R}} \|\mathbf{H}(g,\mathrm{i}\omega)\|_2 \,.$$

One can also consider certain mixed performance measures:

- $\|\mathbf{H}(g,\cdot)\|_{\mathcal{H}_\infty/\mathcal{H}_2}$
- criterion that combines $\|\mathbf{H}(g,\cdot)\|_{\mathcal{H}_2}$ and total average energy [2]

[1] [T./Voigt20]

[2] [Nakić/T./Truhar19]

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Main drawback of direct methods

Damping optimization (position and viscosity optimization):

In the n-mass oscillator:

$$C_{ext} = g(e_i - e_{i+1})(e_i - e_{i+1})^T + g(e_j - e_{j+1})(e_j - e_{j+1})^T,$$

there is a problem with determining optimal (i, j), $1 \leq i \leq j \leq n$ and g.

For example if $n = 1000$:

discrete optimization over 500 000 different damping positions.

Efficient overall algorithm for optimization of damping positions is still needed !

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Consider a parametric LTI dynamical systems represented as

$$\begin{aligned} E\dot{x}(t;p) &= A(p)x(t;p) + Bu(t), \\ y(t;p) &= Cx(t;p), \end{aligned}$$

where $E, A(p) \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{l\times n}$.

- $x(t;p) \in \mathbb{R}^n$ denotes the state variable
- $u(t) \in \mathbb{R}^m$ and $y(t;p) \in \mathbb{R}^l$ represent the inputs and outputs of the system, resp.

We will denote this system with $[\,E, A(p), B, C\,]$.

For parameter p we can approximate our system with reduced system [1]

$$E_r \dot{x}_r(t;p) = A_r(p)x_r(t;p) + B_r u(t;p),$$
$$y_r(t;p) = C_r x_r(t;p),$$

where matrices $V_r \in \mathbb{R}^{n\times r}$ and $W_r \in \mathbb{R}^{n\times r}$ determine reduced system

$$E_r = (W_r)^T E V_r, \qquad A_r = (W_r)^T A V_r,$$
$$B_r = (W_r)^T B \quad \text{and} \quad C_r = C V_r.$$

For set of sampling parameters $p^1, \ldots, p^s$ one can calculate truncation matrices and for global basis we can construct truncation matrices by $V = [V_r^1, \ldots, V_r^s]$ and $W = [W_r^1, \ldots, W_r^s]$.

Problem: reduced order model depends on sampling parameters, but also which sampling one should use .

[1] [Benner/Cohen/Ohlberger/Willcox2017], [Benner/Gugercin/Willcox2015], [Quarteroni/Manzoni/Negri2016], [Quarteroni/Rozza/Manzoni2011]

We would like to remove the need for parametric sampling, which requires identifying particular parameters of interest!

We consider system where $A(p)$ depends on $k \ll n$ parameters $p = (p_1, p_2, \ldots, p_k)$ such that we may write

$$A(p) = A_0 + U \,\operatorname{diag}(p_1, p_2, \ldots, p_k) V^T = A_0 + \sum_{i=1}^{k} p_i u_i v_i,$$

where $U, V \in \mathbb{R}^{n \times k}$ are fixed.

Full-order transfer function

$$\mathbf{H}(s;p) = C(sE - A(p))^{-1} B.$$

Aim: to produce a ROM that retains the structure of parametric dependence and offers uniformly high fidelity across the full parameter range.

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Structure in damping example.

By defining the state-vector $x = [q^T \ \dot{q}^T]^T$ we obtain:

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \dot{x}(t) = A(p)x(t) + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} w(t),$$

$$z(t) = \begin{bmatrix} H_1 & 0 \end{bmatrix} x(t), \quad \text{where}$$

$$A(p) = \begin{bmatrix} 0 & I \\ -K & -C_{int} \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \operatorname{diag}(p_1, p_2, \ldots, p_k) \begin{bmatrix} 0 & B_2^T \end{bmatrix}.$$

Further extensions to the cases with higher rank.

E.g. $A(p) = A_0 + p_1 A_1 + p_2 A_2$ where both A_1, A_2 have rank-2.

Then, one can write $A_1 = [u_1 \ u_2][v_1 \ v_2]^T$ and $A_2 = [u_3 \ u_4][v_3 \ v_4]^T$.
With $U = [u_1 \ u_2 \ u_3 \ u_4]$ and $V = [v_1 \ v_2 \ v_3 \ v_4]$ we obtain

$$A(p) = A_0 + p_1 A_1 + p_2 A_2 = A_0 + U \operatorname{diag}(p_1, p_1, p_2, p_2) V^T.$$

The key observation!

$$\mathbf{H}(s;p) = C\left(\widehat{A}(s) - U\ \mathrm{diag}(p_1, p_2, \ldots, p_k)V^T\right)^{-1} B,\ \widehat{A}(s) = sE - A_0.$$

We use the Sherman-Morrison-Woodbury formula.

$$\mathbf{H}(s;p) = \mathbf{H}_1(s) - \mathbf{H}_2(s)D(p)(I_k + D(p)\mathbf{H}_3(s)D(p))^{-1}D(p)\mathbf{H}_4(s),$$

where parameters are encoded in diagonal matrix $D(p) = \mathrm{diag}(\sqrt{p}_1, \sqrt{p}_2, \ldots, \sqrt{p}_k)$ and

$$\mathbf{H}_1(s) = C\widehat{A}(s)^{-1}B, \qquad \mathbf{H}_2(s) = C\widehat{A}(s)^{-1}U,$$
$$\mathbf{H}_3(s) = V^T\widehat{A}(s)^{-1}U, \qquad \mathbf{H}_4(s) = V^T\widehat{A}(s)^{-1}B.$$

We construct a parameterized reduced order model by using four subsystems which **do not depend on parameters**:

$[E, A_0, B, C]$, $[E, A_0, U, V^T]$, $[E, A_0, U, C]$, and $[E, A_0, B, V^T]$.

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Approach 1: Reduced model based on vector fitting approach

Offline steps

- For the predetermined points in the complex plane $\xi_1, \ldots, \xi_N$ calculate

$$\mathbf{H}_1(\xi_i),\ \mathbf{H}_2(\xi_i),\ \mathbf{H}_3(\xi_i),\ \mathbf{H}_4(\xi_i) \quad \text{for} \quad i = 1, \ldots, N.$$

  These samples do not depend on parameters!

Online steps

- For any given parameter $p = (p_1, p_2, \ldots, p_k)$ calculate $\mathbf{H}(\xi_i; p)$ for $i = 1, \ldots, N$ using obtained formula.
- Based on $\mathbf{H}(\xi_1; p), \ldots, \mathbf{H}(\xi_N; p)$ obtain reduced system with transfer function $\widehat{\mathbf{H}}(s; p)$ using vector fitting approach.

The quality of approximations is determined by

$$e(\mathbf{H}(\cdot; p)), \widehat{\mathbf{H}}(\cdot; p))) = \sum_{i=1}^{N} \left\| \mathbf{H}(\xi_i; p) - \widehat{\mathbf{H}}(\xi_i; p) \right\|_F^2 \Big/ \sum_{i=1}^{N} \|\mathbf{H}(\xi_i; p)\|_F^2.$$

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Approach 2: ROM based on reduction of subsystems

Offline steps

- For underlaying subsystems calculate reduced systems using model reduction techniques for non-parametric systems

$$[\,E, A_0, B, C\,] \to \widehat{\mathbf{H}}_1(s), \text{ using order } r_1;$$
$$[\,E, A_0, U, V^T\,] \to \widehat{\mathbf{H}}_2(s), \text{ using order } r_2;$$
$$[\,E, A_0, U, C\,] \to \widehat{\mathbf{H}}_3(s), \text{ using order } r_3;$$
$$[\,E, A_0, B, V^T\,] \to \widehat{\mathbf{H}}_4(s), \text{ using order } r_4;$$

e. g. using balanced truncation or IRKA approach.

Online steps

- For any given parameter $p = (p_1, p_2, \ldots, p_k)$ obtain approximated system $\widehat{\mathbf{H}}(s; p)$ by

$$\mathbf{H}(s;p) \approx \widehat{\mathbf{H}}_1(s) - \widehat{\mathbf{H}}_2(s) D(p) (I + D(p) \widehat{\mathbf{H}}_3(s) D(p))^{-1} D(p) \widehat{\mathbf{H}}_4(s)$$

Uniform stability of the parameterized reduced model

Theorem

Suppose that the full parameterized model $\mathbf{H}(s, p)$ has been decomposed into subsystems $\mathbf{H}_1(s)$, $\mathbf{H}_2(s)$, and $\mathbf{H}_4(s)$ that are each asymptotically stable, and a subsystem $\mathbf{H}_3(s)$ that is positive real. If the corresponding reduced subsystems $\widehat{\mathbf{H}}_1(s)$, $\widehat{\mathbf{H}}_2(s)$, and $\widehat{\mathbf{H}}_4(s)$ retain asymptotic stability, and $\widehat{\mathbf{H}}_3(s)$ retains positive-realness, then the reduced parameterized model $\widehat{\mathbf{H}}(s, p)$ in

$$\widehat{\mathbf{H}}(s;p) = \widehat{\mathbf{H}}_1(s) - \widehat{\mathbf{H}}_2(s)D(p)(I + D(p)\widehat{\mathbf{H}}_3(s)D(p))^{-1}D(p)\widehat{\mathbf{H}}_4(s).$$

is uniformly asymptotically stable for nonnegative parameters encoded in p.

In order to calculate error bound we consider full order transfer function

$$\mathbf{H}(s;p) = \mathbf{H}_1(s) - \mathbf{H}_2(s)D(p)(I_k + D(p)\mathbf{H}_3(s)D(p))^{-1}D(p)\mathbf{H}_4(s),$$

and corresponding reduced order transfer function

$$\widehat{\mathbf{H}}(s;p) = \widehat{\mathbf{H}}_1(s) - \widehat{\mathbf{H}}_2(s)D(p)(I_k + D(p)\widehat{\mathbf{H}}_3(s)D(p))^{-1}D(p)\widehat{\mathbf{H}}_4(s),$$

we would like to have upper bound for the error

$$\|\mathbf{H}(\cdot;p) - \widehat{\mathbf{H}}(\cdot;p)\| \leq \quad ?$$

Error bound

It can be shown that

$$\mathbf{H}(\cdot;p) - \widehat{\mathbf{H}}(\cdot;p) = [\mathbf{H}_1 - \widehat{\mathbf{H}}_1] + [\widehat{\mathbf{H}}_2 - \mathbf{H}_2]D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)\widehat{\mathbf{H}}_4 + \\ + \mathbf{H}_2 D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)[\widehat{\mathbf{H}}_4 - \mathbf{H}_4] + \\ + \mathbf{H}_2 D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)[\mathbf{H}_3 - \widehat{\mathbf{H}}_3] D(p)\big(I + D(p)\mathbf{H}_3 D(p)\big)^{-1} D(p)\widehat{\mathbf{H}}_4$$

Thus, we have

$$\|\mathbf{H}(\cdot;p) - \widehat{\mathbf{H}}(\cdot;p)\| \leq \|\mathbf{H}_1 - \widehat{\mathbf{H}}_1\| + \|\widehat{\mathbf{H}}_2 - \mathbf{H}_2\| \|D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)\widehat{\mathbf{H}}_4\| + \\ + \|\mathbf{H}_2 D(p)\big(I + D(p)\mathbf{H}_3 D(p)\big)^{-1} D(p)[\widehat{\mathbf{H}}_4 - \mathbf{H}_4]\| + \\ + \|\mathbf{H}_2 D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)\| \|\mathbf{H}_3 - \widehat{\mathbf{H}}_3\| \|D(p)\big(I + D(p)\mathbf{H}_3 D(p)\big)^{-1} D(p)\widehat{\mathbf{H}}_4\|$$

Error bound

It can be shown that

$$\mathbf{H}(\cdot;p) - \widehat{\mathbf{H}}(\cdot;p) = [\mathbf{H}_1 - \widehat{\mathbf{H}}_1] + [\widehat{\mathbf{H}}_2 - \mathbf{H}_2]D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)\widehat{\mathbf{H}}_4 +$$
$$+\mathbf{H}_2 D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)[\widehat{\mathbf{H}}_4 - \mathbf{H}_4] +$$
$$+\mathbf{H}_2 D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)[\mathbf{H}_3 - \widehat{\mathbf{H}}_3] D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)\widehat{\mathbf{H}}_4$$

Thus, we have

$$\|\mathbf{H}(\cdot;p) - \widehat{\mathbf{H}}(\cdot;p)\| \lesssim \|\mathbf{H}_1 - \widehat{\mathbf{H}}_1\| + \|\widehat{\mathbf{H}}_2 - \mathbf{H}_2\| \|D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)\widehat{\mathbf{H}}_4\| +$$
$$+\|\widehat{\mathbf{H}}_2 D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)\| \|[\widehat{\mathbf{H}}_4 - \mathbf{H}_4]\| +$$
$$+\|\widehat{\mathbf{H}}_2 D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)\| \|\mathbf{H}_3 - \widehat{\mathbf{H}}_3\| \|D(p)\big(I + D(p)\widehat{\mathbf{H}}_3 D(p)\big)^{-1} D(p)\widehat{\mathbf{H}}_4\|$$

Which means that we have a bound in terms of

$$\|\mathbf{H}(\cdot;p) - \widehat{\mathbf{H}}(\cdot;p)\| \lesssim \overbrace{\varepsilon_1 + \varepsilon_2 f_1(p, \widehat{\mathbf{H}}_3, \widehat{\mathbf{H}}_4) + \varepsilon_4 f_2(p, \widehat{\mathbf{H}}_2, \widehat{\mathbf{H}}_3) + \varepsilon_3 f_3(p, \widehat{\mathbf{H}}_2, \widehat{\mathbf{H}}_3, \widehat{\mathbf{H}}_4)}^{f(p)}$$

Surrogate optimization with reduced parametric models

A major cost in parameter optimization is the repeated evaluation of the $\mathcal{H}_2$ norm.

We can use the approach 1 or 2 to accelerate computational cost, so we solve a surrogate optimization problem

$$\hat{p}^\star = \arg\min_{p\in\Omega} \left\| \widehat{\mathbf{H}}(\cdot, p) \right\|_{\mathcal{H}_2},$$

where the reduced parametric transfer function $\widehat{\mathbf{H}}(\cdot, p)$ will be constructed using either approach 1 or approach 2, without need for parameter sampling.

Assume $p^\star$ is the minimizer and note that

$$\left\| \mathbf{H}(\cdot, p^\star) \right\|_{\mathcal{H}_2} \leq \left\| \mathbf{H}(\cdot, p^\star) - \widehat{\mathbf{H}}(\cdot, p^\star) \right\|_{\mathcal{H}_2} + \left\| \widehat{\mathbf{H}}(\cdot, p^\star) \right\|_{\mathcal{H}_2}.$$

The surrogate optimization problem will minimize the second term.

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Parameter optimization using reduced models via approach 1

1: Choose the reduced order so that $e(p^0) < \tau$.

2: Solve the surrogate optimization problem

$$\hat{p}^\star = \arg\min_p \left\| \widehat{\mathbf{H}}(\cdot, p) \right\|_{\mathcal{H}_2}$$

with the initial guess p^0 and VF approach for $\widehat{\mathbf{H}}_p$, using $\{\mathbf{H}_i(\xi_i)\}_{i=1}^N$.

3: **while** minimizer $p^\star$ such that $e(p^\star) > \tau$ **do**

4: $p^0 = \hat{p}^\star$

5: Increase the reduced order so that $e(\hat{p}^\star) < \tau$.

6: Determine the new minimizer by solving the

$$\hat{p}^\star = \arg\min_p \left\| \widehat{\mathbf{H}}(\cdot, p) \right\|_{\mathcal{H}_2}$$

using the updated $\widehat{\mathbf{H}}$, the initial guess p^0, and tolerance ν.

7: **end while**

Parameter optimization using reduced models via approach 2

1: Choose the reduced orders r_1, r_2, r_3, r_4 (and $\widehat{\mathbf{H}}_1$, $\widehat{\mathbf{H}}_2$, $\widehat{\mathbf{H}}_3$, $\widehat{\mathbf{H}}_4$) so that $f(p^0) < \tau \, \|\widehat{\mathbf{H}}(\cdot, p^0)\|_{\mathcal{H}_2}$.

2: Solve the surrogate optimization problem

$$\hat{p}^\star = \arg\min_p \left\| \widehat{\mathbf{H}}(\cdot, p) \right\|_{\mathcal{H}_2}$$

with the initial guess p^0 and tolerance ν.

3: **while** minimizer $p^\star$ such that $f(p^\star) > \tau \, \|\widehat{\mathbf{H}}(\cdot, p^0)\|_{\mathcal{H}_2}$ **do**

4: $p^0 = \hat{p}^\star$

5: Increase the orders r_1, r_2, r_3, r_4 s.t. $f(\hat{p}^\star) < \tau \, \|\widehat{\mathbf{H}}(\cdot, p^0)\|_{\mathcal{H}_2}$.

6: Determine the new minimizer by solving the $\hat{p}^\star = \arg\min_p \left\| \widehat{\mathbf{H}}(\cdot, p) \right\|_{\mathcal{H}_2}$ using the updated $\widehat{\mathbf{H}}$, the initial guess p^0, and tolerance ν.

7: **end while**

We consider example from [Penzl 1999]. The full-order system is known and defined by state-space matrices

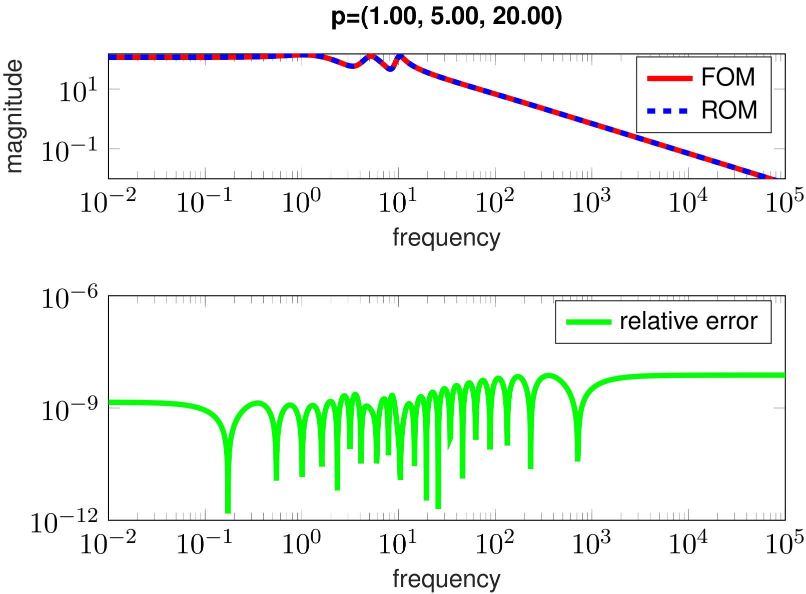
$$A = \operatorname{diag}(A_1(p_1), A_2(p_2), A_3(p_3), -1, -2, \ldots, -N)$$

$$A_i(p_i) = \begin{bmatrix} -1 & p_i \\ -p_i & -1 \end{bmatrix}, \quad \text{for} \quad i = 1, \ldots, 3$$

Matrix $C \in \mathbb{R}^{1\times(N+6)}$ where

$$c_i = \begin{cases} 10, & i = 1, \ldots, 6, \\ 1, & i = 7, \ldots, N. \end{cases}$$

$B = C^T$ and number of states $N = 100$. The parameters p_1, p_2, p_3 represent the imaginary part of the two eigenvalues of the diagonal block $A_i(p_i)$, respectively. Here we use that $p_2 = 5p_1$ and $p_3 = 20p_1$.
We illustrate approach based on balanced truncation of subsystems where four underlaying subsystems were reduced to dimensions 10, 1, 6, 1.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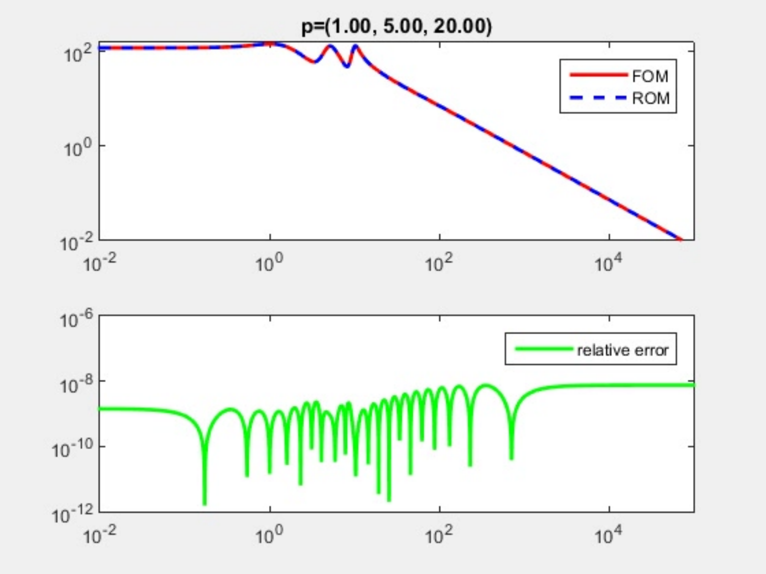
10^{1}
10^{-1}
FOM
ROM
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
frequency
10^{-6}
10^{-9}
10^{-12}
relative error
frequency

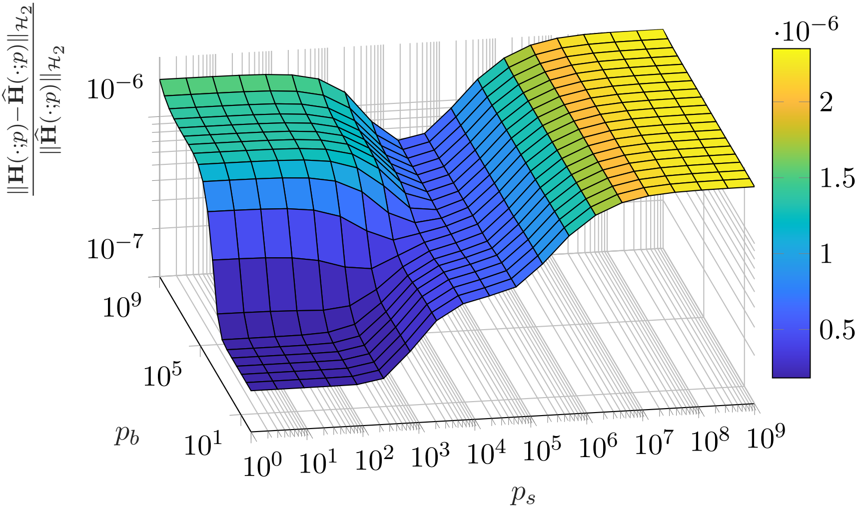

p=(1.00, 5.00, 20.00)
FOM
ROM
relative error
10^{2}
10^{0}
10^{-2}
10^{4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}

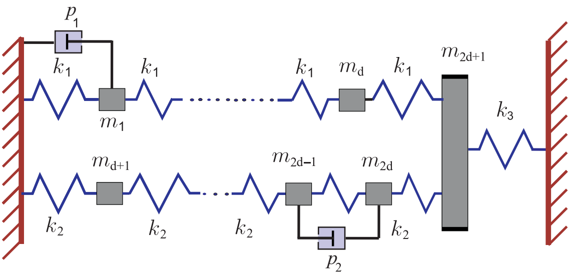
We consider thermal conduction in a semiconductor chip from Oberwolfach Benchmark Collection.
The full-order system is:

$$E\dot{x} = (A - p_t A_t - p_b A_b - p_s A_s)x + Bu$$
$$y = Cx, \quad \text{where}$$

- $E \in \mathbb{R}^{4257 \times 4257}$ corresponds to heat capacity and A to heat conductivity matrix
- $B \in \mathbb{R}^{1 \times 4257}$ is the load vector and $C \in \mathbb{R}^{7 \times 4257}$
- A_t, A_b and A_s are the diagonal matrices from the discretization of the convection boundary conditions with ranks 111, 99 and 31, resp.
- Parameters p_t, p_b, p_s represent film coefficients.

We fix $p_t = 1000$ and vary both p_b and p_s between 1 and 10^9.
Reduced dim. of subsystems: $r_1 = 46, r_2 = 66, r_3 = 200, r_4 = 16$.





The mass and the stiffness matrix are given by

$$K = \begin{bmatrix} K_{11} & & -\kappa_1 \\ & K_{22} & -\kappa_2 \\ -\kappa_1^T & -\kappa_2^T & k_1 + k_2 + k_3 \end{bmatrix}, \quad K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

$\kappa_i = \begin{bmatrix} 0 & \dots & 0 & k_i \end{bmatrix}$ for $i = 1, 2$ and $M = \operatorname{diag}(m_1, m_2, \dots, m_n)$.

$d = 900 \Rightarrow n = 1801$, with $m_{1801} = 1000$ and

$$m_i = \begin{cases} 1000 - \frac{i}{2}, & i = 1, \ldots, 450, \\ i + 325, & i = 451, \ldots, 900, \\ 1300 - \frac{i}{4}, & i = 901, \ldots, n. \end{cases}$$

The stiffness values are given by

$$k_1 = 500, k_2 = 200, k_3 = 300.$$

The primary excitation are 5 disturbances applied to the 4 masses closest to the left-hand side and one mass closest to the right-hand side of oscillator.

We are interested in 2 displacements, i. e.

$$z(t;p) = \begin{bmatrix} q_{400}(t;p) & q_{1300}(t;p) \end{bmatrix}^T.$$

Internal damping is a small multiple of critical damping

$$C_{int} = 0.04 \cdot M^{1/2} \left(M^{-1/2} K M^{-1/2} \right)^{1/2} M^{1/2}.$$

We consider four dampers with gains p_1, p_2, p_3 and p_4 where geometry of positions is given by

$$B_2 = [\, e_{j_1} - e_{j_1+10},\ \ e_{j_2},\ \ e_{j_3},\ \ e_{j_3} - e_{j_3+100} \,],$$

with $j_1 \in \{100, 300, 500, 700\}$, $j_2 \in \{150, 350, 550, 750\}$, $j_3 \in \{1400, 1700\} \Rightarrow$ 32 different damping configurations at which $\| \cdot \|_{\mathcal{H}_2}$ norm was minimized.

Gains were optimized with starting point $p^0 = (100, 100, 100, 100)$ using the full-order model and using proposed reduced systems.

In the approach based on **balanced truncation of subsystems:**

- in all damping configurations, starting reduced dimensions of four subsystems were 280, 300, 480, 430, resp.

In the approach based on **vector fitting approach**

- initial points ξ_i, $i = 1, \ldots, N$, for $N = 500$ depending on modally damped system.
- 130 initial poles (chosen using from dominant poles).

The stoping tolerance for parameter optimization was 0.005.

Time ratio

In average case for one optimization of parameters, new approach **was faster**:

- ≈ 7.8 times, with usage of reduced model based on balanced truncation of subsystems,
- ≈ 60 times, with usage of reduced model based on vector fitting approach.

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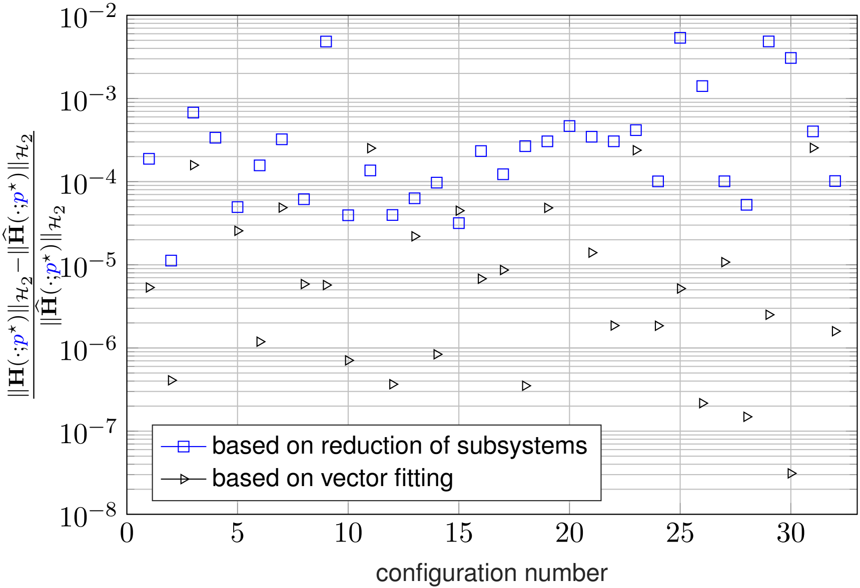
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
$\frac{\|\mathbf{H}(\cdot;p^\star)\|_{\mathcal{H}_2} - \|\widehat{\mathbf{H}}(\cdot;p^\star)\|_{\mathcal{H}_2}}{\|\widehat{\mathbf{H}}(\cdot;p^\star)\|_{\mathcal{H}_2}}$
0
5
10
15
20
25
30
configuration number
based on reduction of subsystems
based on vector fitting

Conclusion :

- We have introduced a framework for producing reduced order models of dynamical systems having an affine, low-rank parametric structure.
  - Approach 1: Reduced model based on vector fitting approach.
  - Approach 2: ROM based on reduction of subsystems.
- The new framework does not require any sampling in the parameter domain and instead parametrically combines intermediate subsystems that are nonparametric.
- Can guarantee uniform stability of the aggregated reduced model across the entire parameter domain in many cases.
- These approaches can be deployed efficiently in parameter optimization problems as well.

Thank you for your attention!