Inverse source problems for coupled heat systems using measurements of one scalar state

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Outline

Inverse problem

Inverse Problem (scalar case)

To establish uniqueness, stability and reconstruction of a heat source $f(x,t)$

$$\begin{cases} u_t - \Delta u = f(x,t) \quad \text{in } \Omega \times (0,T) \\ u(0) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma \times (0,T) \end{cases}$$

from partial data related to u: boundary values (the flux $\partial u / \partial n$ on $\Gamma_0 \subset \partial\Omega$) or internal values ($u$ restricted to $\mathcal{O} \subset \Omega$).

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from partial data related to u: boundary values (the flux $\partial u / \partial n$ on $\Gamma_0 \subset \partial\Omega$) or internal values ($u$ restricted to $\mathcal{O} \subset \Omega$).

Non uniqueness: $u = a(t)\phi(x)$, $\phi \in C^\infty(\Omega)$, $\operatorname{supp} \phi \cap \mathcal{O} = \emptyset$, then $u_t - \Delta u = a'\phi - a\Delta\phi = f$ with $\operatorname{supp} f(\cdot,t) \cap \mathcal{O} = \emptyset$ for each t, so zero measurements in ω.

Some special cases with uniqueness (a priori knowledge)

- Structural identification : $f = f(u)$
  Reaction-diffussion (Cannon-DuChateau 1998, Boulakia-Grandmont-O. 2009, Carleman estimates, Cristofol, Gaitan, Roques, Yamamoto...).
- Indicatrix function : $f = \chi_D$
  (Hettlich-Rundell 2001: domain derivative).
- Punctual support : $f = \sum_{j=1}^{N} p_j \delta_{x_j, t_j}$
  (Yamatani-Ohnaka 1997, El Badia-Ha Duong 2002: backwards heat eqn.).
- When and where it appears? : $f(x_0, T_0) \neq 0$
  (Ikehata 2006: indicatrix functionals).
- Separation of variables : $f = \sigma(t) f(x)$
  (Yamamoto 1995 waves; G.García at al. 2013 heat).

Applications

Identification/validation of sources punctual / stationnary or not

- Pollutant/radiactive/odour emmissions in atmospheric chemistry (at global, regional or megacity scales), *c.f.* Newsam-Enting 1988, Enting 2002, Saide, Bocquet, O., Gallardo 2009.
- Water pollution, coastal, lakes, rivers , *c.f.* Okubo 1980, Linfield 1987.
- Optimal design of monitoring networks , *c.f.* Rodgers 2000, O., Faundez, Gallardo 2013.
- Detection and attribution in climate change , *c.f.* Puel 2002, Garcia, O., Puel 2011, Hannart 2012.
- Detection of phase transition (coupled heat equation) , *c.f.* Homberg, Lu, Sakamoto, Yamamoto, 2013.

Back to the inverse problem

Inverse problem: Given an observatory $\mathcal{O} \subset \Omega$, $T > 0$, if $\sigma(t)$ is known, we want to recover the source $f(x)$ in:

$$\begin{cases} u_t - \Delta u = f(x)\sigma(t) & \text{in} \quad \Omega \times (0,T), \\ u = 0 & \text{on} \quad \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega, \end{cases}$$

from local (in space) measurements of $u|_{\mathcal{O}\times(0,T)}$.

We focus in:

- Uniqueness and stability of $f(x)$ w.r.t. measurements.
- How to design a reconstruction algorithm for $f(x)$ using null controls?

Outline

Null controls?

A null control is a source v with restricted support in that drives the solution of the *backward heat equation* exactly to zero in a given time $\tau > 0$:

$$\begin{cases} -\varphi_t - \Delta\varphi = v|_{\mathcal{O}\times(0,\tau)} & \text{in } \Omega\times(0,\tau) \\ \varphi(\tau) = \varphi_0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma\times(0,\tau). \end{cases}$$

i.e.,

$$\varphi(\tau) = \varphi_0 \quad \text{and} \quad \varphi(0) = 0.$$

It is possible to prove (for instance using global Carleman inequalities [Fursikov-Imanuvilov 1996]) that there exists such a control $v^{(\tau)}$ and

$$\|v^{(\tau)}\|_{L^2(\mathcal{O}\times(0,\tau))} \leq C\exp\left(\frac{C_1}{\tau}\right)\|\varphi_0\|_{L^2(\Omega)}.$$

(this is optimal so this is the "cost" of the null control).

General case

Here, $\sigma \neq cte$ **but known:** To recover $f(x)$ from $u|_{\mathcal{O}\times(0,T)}$ in

$$\begin{cases} u_t - \Delta u = \sigma(t) f(x) & \text{in } \Omega \times (0,T) \\ u(0) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \times (0,T) \end{cases}$$

we take w solution of

$$\begin{cases} w_t - \Delta w = 0 & \text{in } \Omega \times (0,T) \\ w(0) = f(x) & \text{in } \Omega \\ w = 0 & \text{on } \Gamma \times (0,T) \end{cases}$$

and then u defined by

$$u = \int_0^t \sigma(t-\tau) w(\tau) d\tau := Kw$$

satisfies the forward system. Again, we will try to recover f from the following identity (obtained by differentiating u):

$$\sigma(0) w(T) + \int_0^T \sigma'(T-\tau) w(\tau) d\tau = \Delta u(T) + \sigma(T) f(x)$$

Parenthesis: Volterra equation and duality

The definition of K naturally leads to solve a Volterra equation of second kind. Given $v \in L^2(0,T;L^2(\mathcal{O}))$, $\exists! \theta \in H^1(0,T;L^2(\mathcal{O}))$ such that $\theta(T) = 0$ and

$$K^*\theta := \sigma(0)\theta_t + \int_t^T (\sigma(s-t)\theta(s) + \sigma'(s-t)\theta_t(s))\, ds = v(t)$$

with continuous dependence and $\forall w \in L^2(0,T;L^2(\mathcal{O}))$

$$(w, K^*\theta)_{L^2(0,T;L^2(\mathcal{O}))} = (Kw, \theta)_{H^1(0,T,L^2(\mathcal{O}))}.$$

This duality was previously used by [Yamamoto 1995] to derive a source reconstruction formula for the wave equation.

First reconstruction formula

We had

$$\sigma(T)f(x) = -\Delta u(T) + \sigma(0)w(T) + \int_0^T \sigma'(T-\tau)w(\tau)d\tau$$

By introducing the family of null controls $v^{(\tau)}$ controlling from $\varphi(\tau) = \varphi_0$ to $\varphi(0) = 0$ and $K^*\theta^{(\tau)} = v^{(\tau)}$, $Kw = u$ we have

$$\begin{aligned}
&\sigma(T)\int_\Omega f(x)\varphi_0 = \\
&= \ -\int_\Omega \Delta u(T)\varphi_0 + \sigma(0)\int_\Omega w(T)\varphi_0 + \int_0^T \sigma'(T-\tau)\int_\Omega w(\tau)\varphi_0\,d\tau \\
&= -\int_\Omega \Delta u(T)\varphi_0 - \sigma(0)\underbrace{\underbrace{\int_0^T\int_{\mathcal{O}} wv^{(T)}}_{(w,K^*\theta^{(T)})}}_{(u,\theta^T)} - \int_0^T \sigma'(T-\tau)\int_0^T\underbrace{\underbrace{\int_{\mathcal{O}} wv^{(\tau)}\,dt}_{(w,K^*\theta^{(\tau)})}}_{(u,\theta^\tau)}\,d\tau
\end{aligned}$$

Source reconstruction (heat equation)

By observability-controllability duality:

Proposition 1 (García–Osses–Tapia, 2013)

Assume $\sigma \in W^{1,\infty}(0,T)$, $\sigma(T) \neq 0$ *then* $\forall \varphi_0 \in L^2(\Omega)$

$$\int_\Omega f\varphi_0 = -\underbrace{\sigma(T)^{-1}(\Delta u(T), \varphi_0)_{L^2(\Omega)}}_{L} - \underbrace{\sigma(0)\sigma(T)^{-1}(u, \theta^{(T)})_{H^1(L^2(\mathcal{O}))}}_{C_1}$$
$$-\underbrace{\sigma(T)^{-1}\int_0^T \sigma'(T-\tau)(u, \theta^{(\tau)})_{H^1(L^2(\mathcal{O}))}d\tau}_{C_2}$$

where $\theta^{(\tau)}$ *are the solutions of Volterra type associated to null controls* $v^{(\tau)}$ *for* $\tau \in (0,T]$. *Moreover, if* $\sigma'(t) = 0$ *for* $t \in (T-\varepsilon, T]$ *then we can directly obtain*

$$\|f\|_{L^2(\Omega)} \leq C(\|\Delta u(T)\|_{L^2(\Omega)} + \|u\|_{H^1(0,T;L^2(\mathcal{O}))}).$$

Fortunately, it is possible to drop $\Delta u(T)$...

by chosing $\varphi_0 = \varphi_k$ as the eigenfrequencies of the Laplacian.
On one hand:

$$\begin{aligned}\int_\Omega f\varphi_k \quad = \quad & -\underbrace{\sigma(T)^{-1}(\Delta u(T),\varphi_k)_{L^2(\Omega)}}_{L_k} - \underbrace{\sigma(0)\sigma(T)^{-1}(u,\theta_k^{(T)})_{H^1(L^2(\mathcal{O}))}}_{C_{1k}} \\ & -\underbrace{\sigma(T)^{-1}\int_0^T \sigma'(T-\tau)(u,\theta_k^{(\tau)})_{H^1(L^2(\mathcal{O}))}d\tau}_{C_{2k}}\end{aligned}$$

and on the other hand ($\lambda_k > 0$ are the corresponding eigenfrequencies):

$$\int_\Omega f\varphi_k = -\frac{(\Delta u(T),\varphi_k)_{L^2(\Omega)}}{\lambda_k \int_0^T e^{-\lambda_k(T-s)}\sigma(s)ds}$$

so we can eliminate the term in $\Delta u(T)$!

Source reconstruction (heat equation)

Proposition 2 (García–Osses–Tapia, 2013)

Let $f \in L^2(\Omega)$ *and* $\sigma \in W^{1,\infty}(0,T)$, $\sigma(T) \neq 0$ *then*

$$\int_\Omega f\varphi_k = \frac{\mathcal{C}_{1k} + \mathcal{C}_{2k}}{a_k},$$

provided that

$$a_k := 1 - \frac{\lambda_k}{\sigma(T)} \int_0^T e^{-\lambda_k(T-s)}\sigma(s)ds \neq 0,$$

where $\mathcal{C}_{1k}$ *and* $\mathcal{C}_{2k}$ *only depend on measurements* $(u, u_t)|_{\times(0,T)}$.

Remark

If f *is more regular, say* $\|f\|_{D((-\Delta)^\varepsilon)} \leq M$ *for some* $\epsilon \in (0,1)$, *you still have logarithmic conditional stability [García–Takahashi, 2011] [Li-Yamamoto-Zou 2009] :*

$$\|f\|_{L^2(\Omega)} \leq C_{M,\varepsilon} \left|\log \|u_t\|_{L^2(0,T;L^2(\mathcal{O}))}\right|^{-\frac{\varepsilon}{1-\varepsilon}}.$$

Outline

Inverse problem: coupled heat system

Inverse problem: Given an observatory $\mathcal{O} \subset \Omega$, $T > 0$, if $\sigma(t)$ is known, we want to recover the source $F(x) = (f_1(x), f_2(x), \dots, f_n(x))$ in:

$$\begin{cases} \partial_t Y - \Delta Y + QY = \sigma(t)F(x) & \text{in} \quad \Omega \times (0,T), \\ Y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ Y(\cdot, 0) = 0 & \text{in} \quad \Omega, \end{cases}$$

where

$$Q = Q(x) = \begin{pmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1n} \\ 0 & q_{22} & q_{23} & \cdots & q_{2n} \\ 0 & 0 & q_{33} & \cdots & q_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & q_{nn} \end{pmatrix}$$

from local (in space) measurements of $y_i|_{\mathcal{O}\times(0,T)}$, for some $i = 1, \dots, n$.

We focus in:

- Uniqueness and stability of $F(x)$ from local observations.
- How to design a reconstruction algorithm for $F(x)$ using null controls?

A priori knowledge in systems

- Separate variables : $f = \sigma(t)f(x)$
  G.García, – , Osses 2017: Stokes system (source reconstruction).
  Alabau–Boussouira et al. 2016: two wave equations (identification and stability).
- Stability for determining coefficients
  Cristofol at al. 2006 (linear case) and Cristofol at al. 2012 (nonlinear case)(2×2 systems).
  Benabdallah et al. 2009 (2×2 systems).
  Cristofol et al. 2013: discontinuous coefficients (Carleman estimates–optimal control).
  Carreño at al. 2018: hyperbolic systems.
  Dou and Yamamoto 2019: two Schrödinger equations in 3D.

Outline

Case 1: potential matrix with constant coefficients

σ is arbitrary but known: To recover $F(x)$ from $y_n|_{\mathcal{O}\times(0,T)}$ in

$$(1) \qquad \begin{cases} \partial_t Y - \Delta Y + QY = \sigma(t)F(x) & \text{in} \quad \Omega \times (0,T), \\ Y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ Y(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

We take W solution of

$$\begin{cases} \partial_t W - \Delta W + QW = 0 & \text{in} \quad \Omega \times (0,T), \\ W = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ W(\cdot,0) = \sigma(0)F(\cdot) & \text{in} \quad \Omega, \end{cases}$$

and then Y is defined by

$$Y(x,t) = \int_0^t \sigma(s)W(x,t-s)ds =: KW, \quad (x,t) \in \Omega \times (0,T).$$

By evaluating at $t = T$ the main equations of (1), we obtain the following identity:

$$\sigma(0)W(x,T) + \int_0^T \partial_t\sigma(T-s)W(x,s)ds - \Delta Y(x,T) + \int_0^T \sigma(s)QW(x,T-s)ds = \sigma(T)F(x).$$

We had

$$\sigma(0)W(x,T) + \int_0^T \partial_t\sigma(T-s)W(x,s)ds - \Delta Y(x,T) + \int_0^T \sigma(s)QW(x,T-s)ds = \sigma(T)F(x).$$

By multiplying the above identity by elements $\Xi_k := (\varphi_k, \dots, \varphi_k)$ (where $\{\varphi_k\}_{k\in\mathbb{N}}$ are L^2–eigenfunctions of the Laplace operator) and integrating in space, we get

$$\begin{aligned}\sigma(T)(F,\Xi_k)_{L^2(\Omega)^n} =& \sigma(0)(W(x,T),\Xi_k)_{L^2(\Omega)^n} + \int_0^T \partial_t\sigma(T-s)(W(s),\Xi_k)_{L^2(\Omega)^n}ds \\ & - (\Delta Y(T),\Xi_k)_{L^2(\Omega)^n} + \int_0^T \sigma(T-s)(QW(s),\Xi_k)_{L^2(\Omega)^n}ds.\end{aligned}$$

- Consider the decomposition $Y(x,t) = \sum_{k\in\mathbb{N}} Y_k(t)\varphi_k(x)$, where $Y_k(t) = (y_1^k(t), \dots, y_n^k(t))^*$ is the unique solution of the ordinary differential system

(2) $$\left\{\begin{array}{l} Y_k'(t) + (\lambda_k I_n + Q)Y_k(t) = \sigma(t)F_k, \\ Y_k(0) = 0, \end{array}\right.$$

where $F_k = ((f_1,\varphi_k)_{L^2(\Omega)}, \dots, (f_n,\varphi_k)_{L^2(\Omega)})^* =: (f_1^k, \dots, f_n^k)^*$.

- By solving (2), for every $k \in \mathbb{N}$, we obtain

$$Y_k(t) = \overbrace{\left(\int_0^t \tilde{\Phi}_k(t)\tilde{\Phi}_k^{-1}(s)\sigma(s)ds\right)}^{M=(m_{ij}(t))_{i,j=1}^n} F_k = \left(\sum_{j=1}^n m_{1j}(t)f_j^k,\ \sum_{j=1}^n m_{2j}(t)f_j^k,\ \ldots, \sum_{j=1}^n m_{nj}(t)f_j^k\right)^*,$$

where $M = (m_{ij}(t)) = \int_0^t \tilde{\Phi}_k(t)\tilde{\Phi}_k^{-1}(s)\sigma(s)ds$ and $\tilde{\Phi}_k$ is a fundamental matrix associated to the linear ordinary differential system: $Z' + (\lambda_k I_n + Q)Z = 0$.

- Additionally,

$$-(\Delta Y(T), \Xi_k)_{L^2(\Omega)^n} = -(Y(T), \Delta\Xi_k)_{L^2(\Omega)^n} = \lambda_k (Y(T), \Xi_k)_{L^2(\Omega)^n} = \lambda_k \sum_{j=1}^n y_j^k(T).$$

- At this moment, the reconstruction formula is given by:

$$\sum_{j=1}^n \left(1 - \frac{\lambda_k}{\sigma(T)} \sum_{i=1}^n m_{ij}(T)\right) f_j^k = \frac{\sigma(0)}{\sigma(T)}(W(T), \Xi_k)_{L^2(\Omega)^n} + \frac{1}{\sigma(T)} \int_0^T \partial_t \sigma(T-s)(W(s), \Xi_k)_{L^2(\Omega)^n} ds$$
$$+ \frac{1}{\sigma(T)} \int_0^T \sigma(T-s)(QW(s), \Xi_k)_{L^2(\Omega)^n} ds.$$

Null controls with one scalar control?

[M. González–Burgos, L. de Teresa 2010]:
$Q^* \in L^\infty(\Omega)^{n^2}$, $B = diag(0, 0, \ldots, 0, 1), \in \mathcal{M}_n(\mathbb{R})$ and

$$q_{ij} \geq q_0 > 0 \quad \text{in an open set } \mathcal{O}_0 \subset \mathcal{O}, \quad \forall i > j,\, i, j = 1, \ldots, n.$$

Let $\tau \in (0, T]$ and $\Xi_0 \in L^2(\Omega)^n$. Then, there exists a control function $U^{(\tau)} = U^{(\tau)}(\Xi_0) \in L^2(0, T; L^2(\mathcal{O})^n)$ such that the solution Ψ to

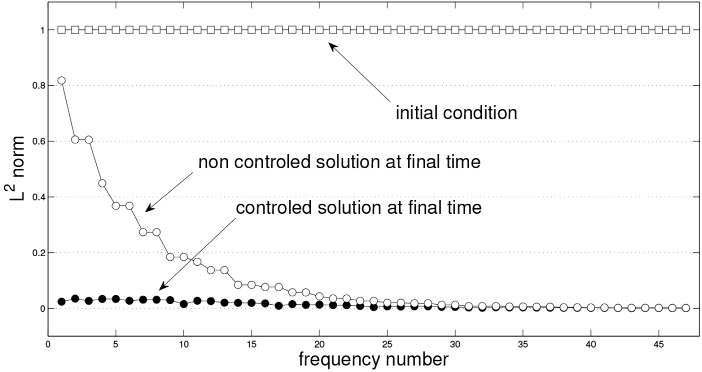
(2)
$$\begin{cases} -\partial_t \Psi - \Delta \Psi + Q^* \Psi = 1_{\mathcal{O}} B U^{(\tau)} & \text{in} \quad \Omega \times (0, \tau), \\ \Psi = 0 & \text{on} \quad \partial\Omega \times (0, \tau), \\ \Psi(\cdot, \tau) = \Xi_0 & \text{in} \quad \Omega, \end{cases}$$

satisfies $\Psi(\cdot, 0) = 0 \quad \text{in } \Omega$.
Moreover, there exists a positive constant C_0 depending only on Ω and $\mathcal{O}$ such that

(3)
$$\|u_n^{(\tau)}\|_{L^2(0,T;L^2(\mathcal{O}))} \leq C_0 e^{C(\tau)} \|\Xi_0\|_{L^2(\Omega)^n}.$$

Now we drive (control) to zero each eigenfrequency starting from a unitary initial condition in L^2

$\Omega = (0,1)^2$; $q_{11} = q_{22} = q_{21} = 1, q_{12} = 0$; $\mathcal{O} = (0,1) \times (0.3,0.7)$;$\Delta t = 5 \times 10^{-3}$; $T = 1$.



- Since Q is a constant matrix, we can solve the following null controllability problems in $\Omega \times (0, s)$:

$$\begin{cases} -\partial_t \Psi - \Delta \Psi + Q^* \Psi = 1_{\mathcal{O}} B U_k^{(s)} \\ \Psi = 0 \\ \Psi(\cdot, s) = \Xi_k(\cdot) \end{cases} \quad \text{and} \quad \begin{cases} -\partial_t \overline{\Psi} - \Delta \overline{\Psi} + Q^* \overline{\Psi} = 1_{\mathcal{O}} Q^* B U_k^{(s)} \\ \Psi = 0 \\ \overline{\Psi}(\cdot, s) = Q^* \Xi_k(\cdot), \end{cases}$$

where $\overline{\Psi} := Q^* \Psi$.
Furthermore, integrating by parts in $L^2(0, s; L^2(\Omega)^n)$, we obtain (after extending $U_k^{(s)}$ by zero at (s, T))

$$(W(s), \Xi_k)_{L^2(\Omega)^n} = -(W, 1_{\mathcal{O}} B U_k^{(s)})_{L^2(0,s;L^2(\Omega)^n)} = -(W, B U_k^{(s)})_{L^2(0,T;L^2(\mathcal{O})^n)}.$$

and

$$(QW(s), \Xi_k)_{L^2(\Omega)^n} = -(W, 1_{\mathcal{O}} Q^* B U_k^{(s)})_{L^2(0,s;L^2(\Omega)^n)} = -(W, Q^* B U_k^{(s)})_{L^2(0,T;L^2(\mathcal{O})^n)}.$$

- Systems of Volterra equations: One system with data $\eta_k^1 := 1_{\mathcal{O}} B U_k^{(s)}$, and another one with data $\eta_k^2 := 1_{\mathcal{O}} Q^* B U_k^{(s)}$. In consequence, we have

$$K^*(\Theta_k^1) = 1_{\mathcal{O}} B U_k^{(s)} \quad \text{and} \quad K^*(\Theta_k^2) = 1_{\mathcal{O}} Q^* B U_k^{(s)}, \quad \forall k \in \mathbb{N}.$$

First reconstruction formula

Theorem 1 (C.M, 2021)

Consider $\sigma \in W^{1,\infty}(0,T)$ *with* $\sigma(T) \neq 0$*. Furthermore, for some* $k \in \mathbb{N}$

(4) $$a_{j,k}^{Q}(T) := \left(1 - \frac{\lambda_k}{\sigma(T)} \sum_{i=1}^{n} m_{ij}(T)\right) \neq 0, \quad \forall i, j = 1, \dots, n,$$

where $M = (m_{ij}(t)) = \int_0^t \tilde{\Phi}_k(t)\tilde{\Phi}_k^{-1}(s)\sigma(s)ds$ *and* $\tilde{\Phi}_k$ *is a fundamental matrix associated to the linear ordinary differential system:* $Z' + (\lambda_k I_n + Q)Z = 0$*. Then, for every solution* $Y \in W_2^{2,1}(\Omega \times (0,T))$ *to* (1)*, the source* $F = (f_1, \dots, f_n)^* \in L^2(\Omega)^n$ *satisfies the local reconstruction identity*

(5) $$\begin{aligned}\sum_{j=1}^{n} a_{j,k}^{Q}(T)(f_j, \varphi_k)_{L^2(\Omega)} = &-\frac{\sigma(0)}{\sigma(T)}(y_n, (\theta_{1,k}^{(T)})_n)_{H^1(0,T;L^2(\mathcal{O}))} \\ &- \frac{1}{\sigma(T)} \int_0^T \partial_t \sigma(T-s)(y_n, (\theta_{1,k}^{(s)})_n)_{H^1(0,T;L^2(\mathcal{O}))} ds \\ &- \frac{1}{\sigma(T)} \int_0^T \sigma(T-s)(y_n, (\theta_{2,k}^{(s)})_n)_{H^1(0,T;L^2(\mathcal{O}))} ds.\end{aligned}$$

Case 2: coupling in the principal part

σ **is arbitrary but known:** To recover $F(x)$ from $y_n|_{\mathcal{O}\times(0,T)}$ in

$$(6) \qquad \begin{cases} \partial_t Y - D\Delta Y = \sigma(t)F(x) & \text{in} \quad \Omega\times(0,T), \\ Y = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ Y(\cdot,0) = 0 & \text{in} \quad \Omega, \end{cases}$$

where the diffusion matrix D^* is diagonalizable with positive real eigenvalues, i.e., for $J = diag(d_i)_{n\times n}$ with $d_1, d_2, \dots, d_n > 0$, one has $D^* = P^{-1}JP$, with $P \in \mathcal{M}_n(\mathbb{R}), detP \neq 0$.
Moreover,

$$d_i \neq d_j, \text{ for } i \neq j,\ 1 \leq i, j \leq n.$$

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Moreover,

$$d_i \neq d_j, \text{ for } i \neq j,\ 1 \leq i, j \leq n.$$

Null controllability property: given an initial datum $\Xi_0 \in L^2(\Omega)^n$, we look for a control function $U \in L^2(0,T; L^2(\mathcal{O})^n)$ such that the corresponding solution Ψ to

$$(7) \qquad \begin{cases} -\partial_t \Psi - D^*\Delta\Psi = 1_{\mathcal{O}}BU & \text{in} \quad \Omega\times(0,T), \\ \Psi = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ \Psi(\cdot,T) = \Xi_0 & \text{in} \quad \Omega, \end{cases}$$

satisfies $\Psi(\cdot,0) = 0$. [Khodja at al 2009, 2011].

Second reconstruction formula

Theorem 2 (C.M, 2021)

Consider $\sigma \in W^{1,\infty}(0,T)$ *with* $\sigma(T) \neq 0$. *Furthermore, for some* $k \in \mathbb{N}$

(8) $$a_{j,k}^{D}(T) := \left(1 - \frac{\lambda_k}{\sigma(T)} \sum_{\ell=1}^{n} \left(\sum_{i=1}^{n} d_{i\ell}\right) m_{\ell j}(T)\right) \neq 0, \quad \forall i,j = 1,\dots,n,$$

where $M = (m_{ij}(t)) = \int\limits_0^t \tilde{\Phi}_k(t)\tilde{\Phi}_k^{-1}(s)\sigma(s)ds$ *and* $\tilde{\Phi}_k$ *a fundamental matrix associated to the ordinary differential system:* $Z' + \lambda_k DZ = 0$.. *Then, for every solution* $Y \in W_2^{2,1}(\Omega \times (0,T))$ *to* (6)*, the source* $F = (f_1,\dots,f_n)^* \in L^2(\Omega)^n$ *satisfies the local reconstruction identity*

(9) $$\begin{aligned}\sum_{j=1}^{n} a_{j,k}^{D}(T)(f_j,\varphi_k)_{L^2(\Omega)} &= -\frac{\sigma(0)}{\sigma(T)}(y_n,(\theta_k)_n)_{H^1(0,T;L^2(\mathcal{O}))} \\ &\quad - \frac{1}{\sigma(T)} \int\limits_0^T \partial_t\sigma(T-s)(y_n,(\theta_k)_n)_{H^1(0,T;L^2(\mathcal{O}))}ds.\end{aligned}$$

Outline

One dimensional model

Inverse problem: To recover $F(x)$ from $y_2|_{\mathcal{O}\times(0,T)}$ in

(10)
$$\begin{cases} \partial_t Y + \overbrace{(-\Delta + Q(x))}^{L} Y = \sigma(t)F(x) & \text{in } (0,\pi)\times(0,T), \\ Y(0,t) = Y(\pi,t) = 0 & \text{in } (0,T), \\ Y(\cdot,0) = 0 & \text{in } (0,\pi), \end{cases}$$

where $L : H^2(0,\pi)^2 \cap H_0^1(0,\pi)^2 \subset L^2(0,\pi)^2 \to L^2(0,\pi)^2$ and Q is given by

$$Q(x) = \begin{pmatrix} 0 & 0 \\ q(x) & 0 \end{pmatrix} \quad \text{and} \quad q \in L^\infty(0,\pi) \cap W^{1,\infty}(\tilde{\mathcal{O}}),\ \tilde{\mathcal{O}} \subset \mathcal{O} \subset (0,\pi).$$

One dimensional model

Inverse problem: To recover $F(x)$ from $y_2|_{\mathcal{O}\times(0,T)}$ in

(10)
$$\begin{cases} \partial_t Y + \overbrace{(-\Delta + Q(x))}^{L} Y = \sigma(t)F(x) & \text{in } (0,\pi)\times(0,T), \\ Y(0,t) = Y(\pi,t) = 0 & \text{in } (0,T), \\ Y(\cdot,0) = 0 & \text{in } (0,\pi), \end{cases}$$

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- Consider the families (for $k \in \mathbb{N}$) (here, φ_k are the normalized eigenfunctions of the Laplace operator)

$$\mathcal{B} = \left\{ \Phi_{1,k} = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \Phi_{2,k} = \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} \right\} \text{ and } \mathcal{B}^* = \left\{ \Phi^*_{1,k} = \begin{pmatrix} \psi_k \\ \varphi_k \end{pmatrix}, \Phi^*_{2,k} = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\},$$

(11)
$$\begin{cases} \psi_k(x) = \alpha_k\varphi_k(x) - \dfrac{1}{k}\displaystyle\int_0^x \sin(k(x-\zeta))(I_k(q)\varphi_k(\zeta) - q(\zeta)\varphi_k(\zeta))d\zeta;\ I_k(q) := \int_0^\pi q(x)\varphi_k(x)dx, \\ \alpha_k = \dfrac{1}{k}\displaystyle\int_0^\pi\int_0^x \sin(k(x-\zeta))((I_k(q)\varphi_k(\zeta) - q(\zeta)\varphi_k(\zeta)))\varphi_k(x)d\zeta dx. \end{cases}$$

Spectral analysis

Then, one has [Duprez 2017]

a) The spectrum of L^* and L are given by $\rho(L^*) = \rho(L) = \{k^2 : k \in \mathbb{N}\}$.

b) For every $k \in \mathbb{N}$, the eigenvalue k^2 of L^* has algebraic multiplicity 1. Moreover, in this case,

$$\left\{ \begin{array}{l} \left(L^* - k^2 Id\right) \Phi^*_{1,k} = I_k(q)\Phi^*_{2,k}, \\ \left(L^* - k^2 Id\right) \Phi^*_{2,k} = 0. \end{array} \right. \tag{12}$$

c) For every $k \in \mathbb{N}$, the eigenvalue k^2 of L has algebraic multiplicity 1. Moreover, in this case,

$$\left\{ \begin{array}{l} \left(L - k^2 Id\right) \Phi_{1,k} = 0, \\ \left(L - k^2 Id\right) \Phi_{2,k} = I_k(q)\Phi_{1,k}. \end{array} \right. \tag{13}$$

d) The sequences $\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal Riesz basis of $L^2(0,\pi)^2$.

e) The sequence $\mathcal{B}^*$ is a Schauder basis of $H^1_0(0,\pi)^2$ and $\mathcal{B}$ is its biorthogonal basis in $H^{-1}(0,\pi)^2$.

Third reconstruction formula

H1 Consider $\sigma \in W^{1,\infty}(0,T)$ with $\sigma(T) \neq 0$. Furthermore, for some $k \in \mathbb{N}$

$$a_k^L(T) := \sigma(T)\left(1 - \frac{k^2}{\sigma(T)}\int_0^T e^{-k^2(T-s)}\sigma(s)ds\right) \neq 0,$$
$$b_k^L(T) := -I_k(q)\left(1 - k^2\int_0^T (T-s)e^{-k^2(T-s)}\sigma(s)ds\right).$$

H2 Consider the above result concerning spectral analysis, see [Duprez 2017].

H3 Null controllability. For any $s \in (0,T]$, assume that the adjoint system associated to (34) with distributed control $1_{\mathcal{O}}U^{(s)} = (0, 1_{\mathcal{O}}u_2^{(s)})$ satisfies the null controllability property; [Duprez 2017].

H4 Consider Volterra equations and its properties.

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H4 Consider Volterra equations and its properties.

Theorem 3 (C.M, 2021)

Let H1–H4 be satisfied. Then, for any solution $Y \in W_2^{2,1}((0,\pi)\times(0,T))$ *of* (34)*, the source* $F = (f_1, f_2) \in L^2(0,\pi)^2$ *satisfies*

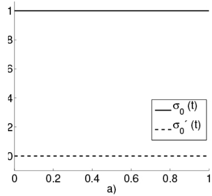
(14)
$$a_k^L(T)\left(f_1^{\varphi_k} + f_1^{\psi_k} + f_2^{\varphi_k}\right) + b_k^L(T) f_1^{\varphi_k} = -\sigma(0)(y_2, \theta_k^{(s)})_{H^1(0,T;L^2(\mathcal{O}))} - \int_0^T \partial_t \sigma(T-s)(y_2, \theta_k^{(s)})_{H^1(0,T;L^2(\mathcal{O}))} ds,$$

where $f_1^{\varphi_k} := (f_1, \varphi_k)_{L^2(0,\pi)}$, $f_1^{\psi_k} := (f_1, \psi_k)_{L^2(0,\pi)}$ *and* $f_2^{\varphi_k} := (f_2, \varphi_k)_{L^2(0,\pi)}$.

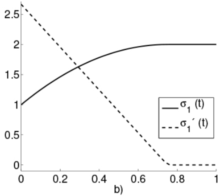
Outline

Three cases of $\sigma(t)$

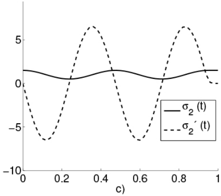
a) $\sigma = \sigma_0$ constant.
b) σ non negative and increasing.
c) $\sigma = \sigma_0 + b\cos(\omega t)$, σ_0 constant, $b \neq 0$, y $\omega \in \mathbb{R} \setminus D$, where D is an appropriate discrete set.



case a)



case b)



case c)

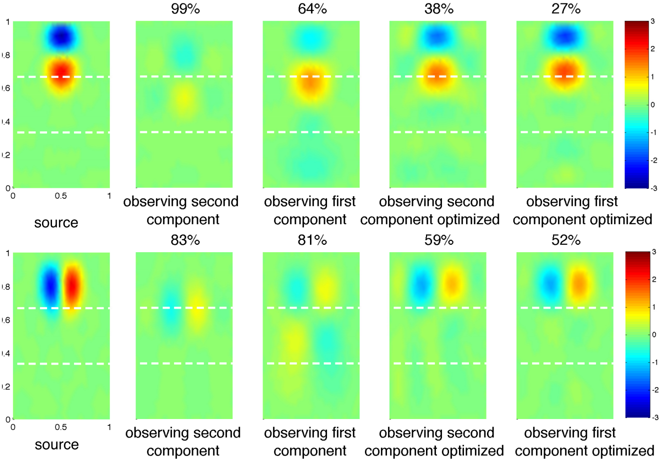
Example in 2D

Consider $\mathcal{O} = (0,1) \times (0.3,0.7)$ and $T = 1$.
The inverse problem consists in recovering $F(x_{1,x_2}) = (f_1(x_1,x_2), f_1(x_1,x_2))$ from $y_2|_{\mathcal{O}\times(0,T)}$ in

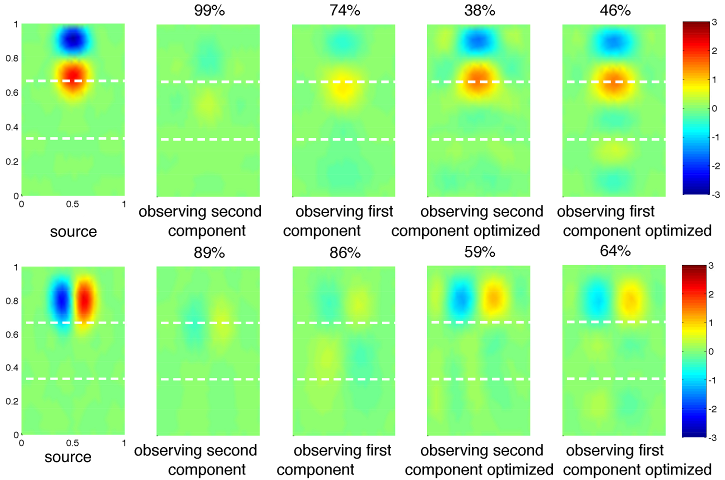
$$\begin{cases} \partial_t y_1 - \Delta y_1 + y_1 + y_2 = \sigma(t) f_1(x_1,x_2) & \text{in } (0,1)^2 \times (0,T), \\ \partial_t y_2 - \Delta y_2 + y_2 = \sigma(t) f_2(x_1,x_2) & \text{in } (0,1)^2 \times (0,T), \\ (y_1,y_2)(0,t) = (y_1,y_2)(1,t) = 0 & \text{in } (0,T), \\ (y_1,y_2)(\cdot,0) = 0 & \text{in } (0,1). \end{cases}$$

Can we observe the first component $y_1|_{\mathcal{O}\times(0,T)}$?

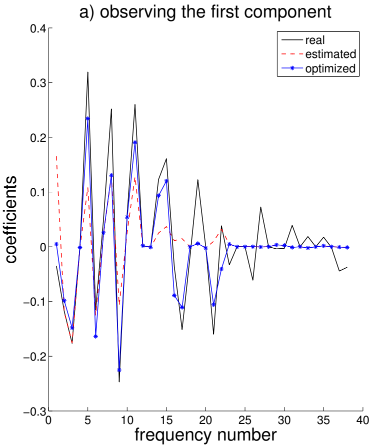
$\Omega = (0,1)^2$, $\mathcal{O} = (0,1) \times (0.3{,}0.7)$. 5 % noise. 40 freq. Gaussian sources - σ oscillating

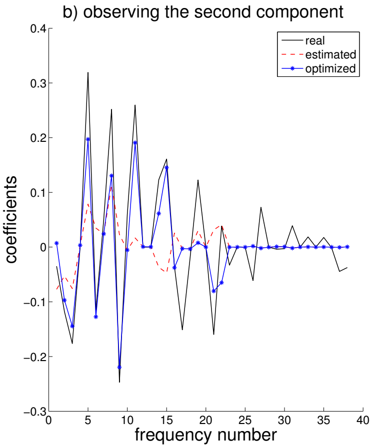


$\Omega = (0,1)^2$, $\mathcal{O} = (0,1) \times (0.3,0.7)$. 5 % noise. 40 freq. Gaussian sources - σ increasing



Source coefficients - case σ increasing





Comments and open problems

- Reconstruction formulas involving controllability to zero, Volterra equations, spectral analysis. Finite elements, Hilbert Uniqueness Method (HUM).
- Numerical experiments: coupling in the main operator & model in 1D; more examples.
- Is it possible to extend the above strategy to nonlinear cases?
- Can one extend the model in 1D to higher dimensions?
- Can one apply this framework to dispersive models (i.e., Korteweg–de Vries linear equation)?, flame models (Kuramoto–Sivanshinky equation)?
- Can we optimize the above procedure?

Hvala vam puno

Thank you very much