Cost of the Null-Controllability of Parabolic Partial Differential Equations

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Motivatinng questions

How large is the control cost of the null-controllability?

How does it scale with parameters of the system?

Can we recover phyically reasonable relations in the control cost? E.g. "space parameter"^{2} $\sim$ "time" for control of the heat equation $\dot{w} = \Delta w$.

Outline

What is controllability?

Null-controllability with explicit cost.

Scalable unique continuation principle for $A = -\Delta + V$.

Putting things together.

Setting

X, Y Hilbert spaces. A self-adjoint in X, $A \geq \kappa$, $\kappa \in \mathbb{R}$.
$B : Y \to X$ bounded.

The equation in the unknown $w : [0, T] \to X$

$$\dot{w} = Aw + Bu, \quad w(0) = w_0.$$

u is called the control function, the solution w is called the state.
This equation models a control system.
The solution is given by

$$w(t) = e^{tA} w_0 + \int_0^t e^{(t-s)A} Bu(s)\, ds, \text{ for all } 0 \leq t \leq T.$$

Remark: B can be chosen as (in a certain sense) unbounded. We will not focus on that.

Concepts of controllability

Definition

The control system is exactly controllable in time T if for any w_0, w_T there exists u such that the state w satisfies $w(T) = w_T$.

The control system is null controllable in time T if for any w_0 there exists u such that the state satisfies $w(T) = 0$.

The control system is approximatively controllable in time T if for any w_0, w_T and any $\varepsilon > 0$ there exists u such that the state w satisfies $\|w(T) - w_T\| < \varepsilon$.

Null controllability test

Theorem

The system is null controllable in time T if and only if there exists a constant c > 0 *such that*

$$\int_0^T \|B^* e^{tA^*} \varphi_0\|^2 \, dt \geq c \|e^{TA^*} \varphi_0\|, \text{ for all } \varphi_0.$$

This is called an observability inequality.

Null-controllability

$$\dot{w} + Aw = Bu, \quad w(0) \in X, \quad A \geq \kappa.$$

Theorem (Sufficient condition for null-controllability, NTTV '18)

Let $d_0 \geq 1$, $d_1 > 0$, such that for all $\phi \in X$, $\lambda \geq 0$:

$$\|P_A(-\infty, \lambda)\phi\|_X^2 \leq d_0 e^{d_1 \sqrt{\lambda}} \|B^* P_A(-\infty, \lambda)\phi\|_Y^2.$$

Then, for all $T > 0$, $w(0) \in X$, there is an input function u such that $w(T) = 0$, and

$$\int_0^T \|u(t)\|^2 \mathrm{d}T \leq C_{\mathrm{control}} \|w(0)\|_X^2,$$

AND we know a lot about $C_{\mathrm{control}} = C_{\mathrm{control}}(T, d_0, d_1, B, \kappa)$.

(Details on C_{control} in a minute!)

What's new?

We simultaniously treat:

- A with not necessarily discrete spectrum (e.g. full space Laplacian),
- A not necessarily positive (e.g. $A = -\Delta + V$ Schrödinger operator),
- all $T > 0$,
- possibly unbounded input operator B (e.g. for boundary control),
- explicit dependence of C_{control} on T, d_0, d_1, $\|B\|$, κ.
- Optimal or (as far as we know) the best known w.t.r. to the existing literature.

Every single item on this list has somewhere already been treated, but the combination (to our knowledge) not.

How does C_{control} look like? (1)

Let us first give some clues.

<u>If $T \searrow 0$:</u>

$$C_{\text{control}} \sim \exp\left(\frac{C}{T}\right). \quad \text{Sharp.}$$

<u>If $T \nearrow \infty$:</u>

Recall $A \geq \kappa$. The free system satisfies $w(t) = e^{-At} w(0)$.

If $\kappa > 0$: $C_{\text{control}} \sim e^{-C\kappa}$. "Large times are your friend." Sharp.

If $\kappa = 0$: $C_{\text{control}} \sim T^{-1}$. "Large times are still your friend."

If $\kappa < 0$: $C_{\text{control}} \geq C > 0$. "Large times don't really help you."

How does C_{control} look like? (2)

Dependence on d_0, d_1:

Recall the assumption was:

$$\left\|P_A(-\infty,\lambda)\phi\right\|_X^2 \le d_0 e^{d_1\sqrt{\lambda}}\left\|B^* P_A(-\infty,\lambda)\phi\right\|_Y^2, \quad d_0 \ge 1, d_1 > 0.$$

$C_{\text{control}} \sim \exp(Cd_1^2)$. Sharp.

$C_{\text{control}} \sim d_0 \exp(C\log(d_0)^2)$. Subexponential in d_0.

Here it is:

$$C_{\text{control}} = C\frac{d_0}{T}\exp\left(\frac{Cd_1^2}{T} + \frac{C\ln^2(d_0(\|B\|^2+1))}{\min\{T^2;1\}} - 2\kappa T\right)$$

So far ...

... we have seen that it is nice to have an inequality

$$\left\|P_A(-\infty,\lambda)\phi\right\|^2 \leq d_0 \mathrm{e}^{d_1\sqrt{\lambda}}\left\|B^* P_A(-\infty,\lambda)\phi\right\|_Y^2$$

and to know as much as possible about d_0 and d_1.

Scalable quantitative unique continuation principle

Theorem (N., Täufer, Tautenhahn, Veselić '18)
Let $G > 0$, $\delta < G/2$ and $S_{\delta,G}$ a union of δ-balls in $\mathbb{R}^d$ such that every elementary cell of $G \cdot \mathbb{Z}^d$ contains exactly one δ-ball.

Let $V \in L^\infty(\mathbb{R}^d)$.

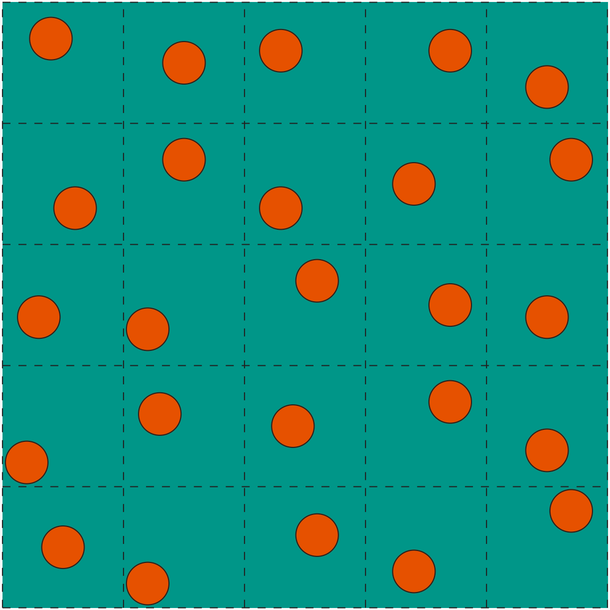
Then, for every $\phi \in L^2(\mathbb{R}^d)$, $\lambda \geq 0$, we have

$$\|P_{-\Delta+V}(-\infty,\lambda)\phi\|^2_{L^2(\mathbb{R}^d)} \leq d_0 e^{d_1\sqrt{\lambda}} \|\chi_{S_{\delta,G}} \cdot P_{-\Delta+V}(-\infty,\lambda)\phi\|^2_{L^2(\mathbb{R}^d)},$$

where

$$d_0 = (\delta/G)^{C(1+G^{4/3}\|V\|_\infty^{2/3})}, \quad d_1 = C\ln(\delta/G)G.$$

Geometric setting



Some remarks

Our system is a heat system with a heat generation term V, and where the control function acts on the set $S_{\delta,G}$:

$$\dot{w} = (-\Delta + V)w + \chi_{S_{\delta,G}} u.$$

- The theorem generalizes in a natural way from $\mathbb{R}^d$ to generalized rectangles (finite cubes, half-spaces, slabs in $\mathbb{R}^d$, ...).
- Instead of Δ we can have a 2nd order elliptic differential operator.
- Proof relies on Carleman estimates.
- We also have a result for general sets, not just balls, but for $V = 0$.

Putting things together

Let $V \in L^\infty(\mathbb{R}^d)$, G, δ, $S_{\delta,G}$ as before, and $T > 0$. Then

$$\partial_t \phi(x,t) - \Delta\phi(x,t) + V(x)\phi(x,t) = \chi_{S_{G,\delta}}(x)u(x,t) \quad \text{in } L^2(\mathbb{R}^d \times [0,T])$$

is null-controllable with cost

$$C_{\text{control}} = C\frac{d_0}{T}\exp\left(\frac{Cd_1^2}{T} + \frac{C\ln^2(d_0)}{\min\{T^2;1\}} - 2\|V\|_\infty T\right)$$

where

$$d_0 = \left(\frac{\delta}{G}\right)^{C(1+G^{4/3}\|V\|_\infty^{2/3})}, \quad d_1 = C\ln(\delta/G)G.$$

This still looks a bit too complicated for the moment!

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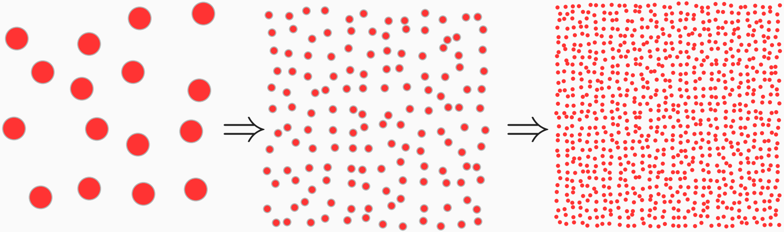
So let us put $V = 0$ for simplicity and keep δ/G constant.

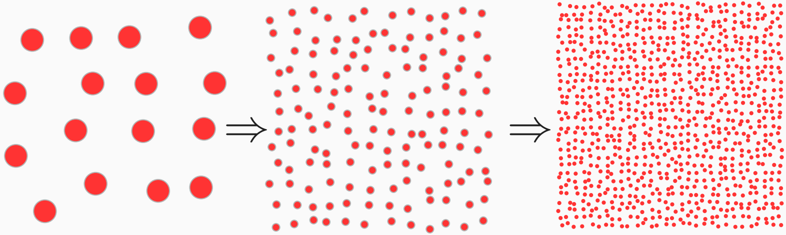
Homogenization



Let us send $G \searrow 0$ while keeping δ/G constant.

$$C_{\text{control}} = \frac{C}{T}\exp\left(C\frac{G^2}{T} + C\right).$$

Homogenization

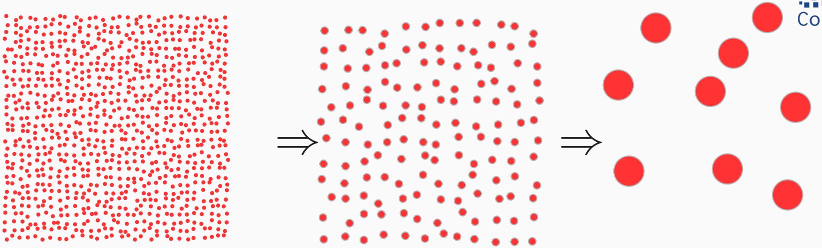


Let us send $G \searrow 0$ while keeping δ/G constant.

$$C_{\text{control}} = \frac{C}{T} \exp\left(C\frac{G^2}{T} + C \right).$$

Control set becomes more and more equidistributed.
Control cost improves and converges to something.
Maybe interesting.

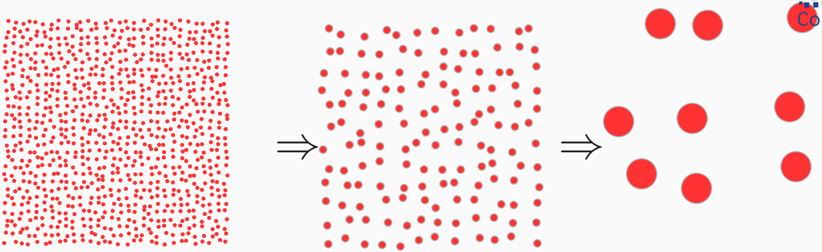
De-homogenization



Let us send $G \to \infty$ while keeping δ/G constant.

$$C_{\text{control}} = \frac{C}{T} \exp\left(C\frac{G^2}{T} + C \right).$$

De-homogenization



Let us send $G \to \infty$ while keeping δ / G constant.

$$C_{\text{control}} = \frac{C}{T} \exp\left(c\frac{G^2}{T} + C \right).$$

Diverges when $G \nearrow \infty$.
This can be accomodated by giving more time, i.e. $T \sim G^2$.
Compare $\Delta \sim \partial_t$ in heat equation to $G^2 \sim T$ in control cost. Same relation as dimensions of time and space derivatives in the PDE. ☺

Thanks for the attention!

ConDyS