Optimal control of parabolic equations using spectral calculus

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The problem

Inital condition or starting optimal control: solve the following problem

$$\min_{u \in \mathcal{H}} \{J(u) \colon \|y(T) - y^*\| \leq \varepsilon\},$$

where (in the weak sense)

$$\begin{cases} y'(t) + Ay(t) = f(t) & \text{for } 0 \leq t \leq T, \\ y(0) = u, \end{cases}$$

$$J(u) = \frac{\alpha}{2}\|u\|^2 + \frac{1}{2}\int_0^T \beta(t)\|y(t) - w(t)\|^2 \, \mathrm{d}t.$$

Here we assume: A selfadjoint, lower semi-bounded operator on a Hilbert space $\mathcal{H}$, $f \in L^2((0,T);\mathcal{H})$. Parameters: y^* is the target state, $\varepsilon > 0$ is the tolerance, $\alpha > 0$, $\beta \in L^\infty((0,T);[0,\infty))$ are weights, and $w \in L^2((0,T);\mathcal{H})$ is the desired trajectory of the system.

The solution

The solution $\hat{u}$ of the problem is given by

$$S_T\hat{u} = (\hat{\mu} + B)^{-1}(\hat{\mu} y_h^* + b) - y^* - y_h^*,$$

where

$$B = \alpha I + \int_0^T \beta(t) S_{2t}\, \mathrm{d}t, \quad b = \int_0^T \beta(t) S_{T+t} w_h(t)\, \mathrm{d}t,$$

$$y_h^* = y^* - \int_0^T S_\tau f(\tau)\, \mathrm{d}\tau, \quad w_h = w - \int_0^{\cdot} S_\tau f(\tau)\, \mathrm{d}\tau,$$

$\hat{\mu}$ is the unique solution of

$$G(\mu) := \|y_h^* - (\mu + B)^{-1}(\mu y_h^* + b)\| = \varepsilon,$$

and $\{S_t\}$ is the semigroup generated by $-A$.

Remarks

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- → The function G is decreasing.
- → Let $g(\mu) = y_h^* - (\mu + B)^{-1}(\mu y_h^* + b)$, so $G(\mu) = \|g(\mu)\|$. Then $g(\mu) = y_h^* - x$, where x is the solution of the equation

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- The optimal final state is the solution of the equation

$$(\hat{\mu} + B)\, x = \hat{\mu} y_h^* + b,$$

  hence we obtain it for free.

Remarks

→ In applications, it is enough to find good μ ($\mu \geq \hat{\mu}$, μ close to $\hat{\mu}$), not the optimal one. One choice is to take

$$\mu = \frac{\|By_h^* - b\|}{\varepsilon} + \alpha + \int_0^T \frac{1}{2t}\beta(t)\mathrm{e}^{-2t\kappa}\,\mathrm{d}t,$$

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- → In applications, we can use $B = \tilde{\beta}_0(A)$, where $\tilde{\beta}_0(\lambda) = \alpha + \int_0^T \beta(t)\exp(-2t\lambda)\,\mathrm{d}t$ and we can approximate b by ($w_h(t) \approx \sum_{i=1}^N w_i \chi_{[t_{i-1},t_i]}$)

$$\sum_{i=1}^N \tilde{\beta}_i(A) w_i, \text{ where } \tilde{\beta}_i(\lambda) = \int_{t_{i-1}}^{t_i} \beta(t)\exp(-(T+t)\lambda)\,\mathrm{d}t,$$

An example

Let A be positive definite operator and let $f = 0$. We take $\beta = \chi_{[T/3, 2T/3]}$ and assume that w does not depend on time.
Then

$$B = \alpha + \frac{1}{2} A^{-1} S_{2T/3}(I - S_{2T/3}),$$
$$b = A^{-1} S_{4T/3}(I - S_{T/3}) w.$$

Sensitivity of the problem

Let us perturbe all the parameters of the problem with perturbations $< \nu$, in respective norms, such that that the perturbed problem has the same structure (*A* still selfadjoint etc.). Then

Theorem

For small enough $\nu > 0$ the optimal solutions of the original and the perturbed problem differ by $< C\nu$ with explicit C.

Idea of the proof

Proof is mostly geometry.
We can assume $f = 0$. We work in $\tilde{\mathcal{H}} = \operatorname{Ran} S_T$, with the scalar product $\langle S_T^{-1}\cdot, S_T^{-1}\cdot\rangle$, and define

$$\omega(\cdot) = J(S_T^{-1}\cdot).$$

Then $\hat{y}$ is the unique solution of

$$\min_{y\in\tilde{\mathcal{H}}} \{\omega(y) \colon \|y - y^*\| \leq \varepsilon\}\,.$$

We define

$$W_c = \{y \in \tilde{\mathcal{H}} \colon \omega(y) \leq c\}.$$

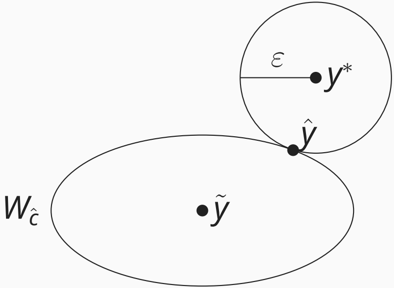
Let $\Pi_c(x)$ be a projection of x to W_c. Then

$$\Pi_{\hat{c}}(y^*) = \hat{y}, \text{ where } \hat{c} = \omega(\hat{y}).$$

Idea of the proof

The second geometric ingredient is the following result:

there exists $\hat{\gamma} > 0$ such that $y^* - \hat{y} = \hat{\gamma} \nabla \omega(\hat{y})$.



Non-homogeneous boundary condition

Suppose we have

$$\begin{cases} y'(t) + Ly(t) = 0 & \text{for } 0 \leq t \leq T, \\ Gy(t) = g(t), \\ y(0) = u, \end{cases}$$

where G is a boundary trace operator.
We assume that (L, G) forms so-called boundary control system.

Definition

A boundary control system is a pair of operators (L, G) where $L \in L(Z, X)$ and $G \in L(Z, U)$, if there exists $\beta \in \mathbb{C}$ such that:
G is surjective, Ker G is dense in X, $\beta - L$ restricted to Ker G is surjective, and $\mathrm{Ker}(\beta - L) \cap \mathrm{Ker}\, G = \{0\}$.

Non-homogeneous boundary condition

We define the operator A on X by $Au = Lu$ and $D(A) = \operatorname{Ker} G$. Let X_{-1} be the extrapolation space corresponding to A and let $\hat{A}$ be the extension of A to X_{-1}. There exists a unique $T \in L(U, X_{-1})$ such that the problem can be written as

$$\dot{y}(t) + \hat{A}y(t) = Tg(t).$$

So we are back in business if A is a lower semi-bounded selfadjoint operator ($\hat{A}$ inherits the properties of A) but X_{-1} is not a nice space to work with.

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Fear not, B and b can still be seen as operator/element in X using the fact that $Tg(\cdot) = (\beta - \hat{A})h(\cdot)$, for h function with values in X.
If g is constant in time, B and b have nice formulas.

What's the point?

Standard (and much more general) solution is based on Lagrange multipliers, and to find the solution one needs to solve two coupled time-dependent problems, the original and the adjoint one.
We need to solve just one stationary problem, but with a more complicated operator and vector.

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Should have some advantages. Constructing efficient numerical procedure is a work in progress. To learn about one possible approach go to Luka’s talk.

Look ahead

→ proper numerics

Look ahead

→ proper numerics
→ distributed control

Look ahead

→ proper numerics
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→ boundary control

Look ahead

- → proper numerics
- → distributed control
- → boundary control
- → non-selfadjoint case

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Thanks for the attention!