Structured Lyapunov Functions and Dissipativity in LTI dynamical networks

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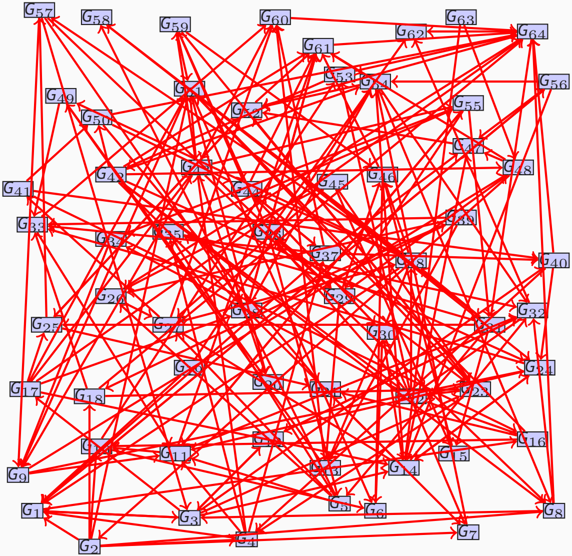
Joint work with A. Jokić   arXiv:1905.05854

IWOTA 2019, Lisbon, July 23, 2019

Financed by the HRZZ project IP-2016-06-2468

LTI dynamical networks

An LTI dynamical network



A representation of an LTI dynamical network

For the moment we assume that the dynamical network is *closed*, i.e. there are no exogenous inputs and outputs.
Dynamical network is a directed graph $\Gamma = (\Omega, E)$.
Each vertex $G^i \in \Omega$ is a linear time invariant (LTI) dynamical system.
Directed edge $(G^i, G^j) \in E$ implies that the dynamics of the system G^i influences the dynamics of the system G^j, i.e. there is an output signal of G^i that is input to G^j.

Let w^{ij} be the output from G^i that is the input of G^j, v^{ij} be the input of G^i from G^j. Obviously $v^{ij} = w^{ji}$.
Let $v^i = \text{col}(v^{ij})$, $w_i = \text{col}(w^{ij})$. The system G^i is given by

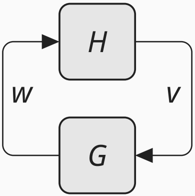
$$\begin{pmatrix} \dot{x}^i \\ w^i \end{pmatrix} = \begin{pmatrix} A^i & B^i \\ C^i & D^i \end{pmatrix} \begin{pmatrix} x^i \\ v^i \end{pmatrix}$$

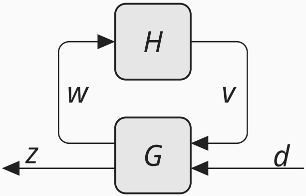
A representation of an LTI dynamical network

Let $v = \mathrm{col}(v^i)$, $w = \mathrm{col}(w^i)$, $x = \mathrm{col}(x^i)$, $A = \mathrm{diag}(A^i)$,
$B = \mathrm{diag}(B^i)$, $C = \mathrm{diag}(C^i)$, $D = \mathrm{diag}(D^i)$.
Then the dynamical network is described by

$$\begin{pmatrix} \dot{x} \\ w \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix},$$
$$v = Hw,$$

where H encodes the topology of the dynamical network.





Dissipativity theory

Dissipative systems

A dynamical system $\dot{x} = f(x, d)$, $z = g(x, d)$, is dissipative with respect to the supply function (rate) $s(\cdot, \cdot)$ if there exists a storage $V : X \to \mathbb{R}$ such that the dissipation inequality

$$V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} s(d(t), z(t)) \, \mathrm{d}\, t$$

holds for all trajectories of x, d, z satisfying the system's dynamics, for all $t \in [t_0, t_1]$ and for all $t_0 < t_1$.
The dynamical system is said to be strictly dissipative if there exists $\epsilon > 0$ so that the dissipation inequality holds when $s(d(t), z(t))$ is replaced with $s(d(t), z(t)) - \epsilon \|d(t)\|^2$.

supply: power supplied to (or extracted from) the system

storage: energy stored within the system (Lyapunov function)

dissipation inequality: difference in the stored energy cannot exceed the amount of energy supplied to the system

Quadratic supply function

An LTI system G is strictly dissipative with respect to the quadratic supply function

$$s(d,z) = \begin{pmatrix} d \\ z \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} d \\ z \end{pmatrix},$$

where Q and R are symmetric matrices, if there exists a quadratic storage function

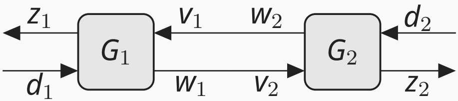
$$V(x) = x^\top P x,$$

such that the time derivative of $V(x(t))$ along the system's trajectories satisfies the *strict differential dissipation inequality*

$$\dot{V}(x(t)) < s(d(t), z(t))$$

at any time t and for all $x \neq 0$, $d \neq 0$, z related via the system G.

Interconnection neutral supply rates



Let G_i be dissipative with respect to the supply function $s_i(v_i, d_i, w_i, z_i)$ with a storage function $V_i(x_i)$ and assume that the supply functions have the following additive structure

$$s_i(v_i, d_i, w_i, z_i) = s_{i,\text{ext}}(d_i, z_i) + s_{i,\text{int}}(v_i, w_i), \qquad i = 1, 2.$$

The interconnection is said to be *neutral* with respect to the supply functions $s_{1,\text{int}}$, $s_{2,\text{int}}$ if the neutrality condition

$$s_{1,\text{int}}(v_1, w_1) + s_{2,\text{int}}(v_2, w_2) = 0,$$

holds for all v_1, w_1, v_2, w_2 such that $v_1 = w_2$, $v_2 = w_1$.

Interconnection neutral supply rates

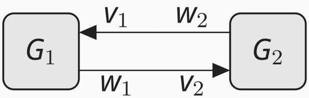
We will use the term interconnection neutral supply functions to refer to supply functions $s_{1,\text{int}}(v_1, w_1)$ and $s_{2,\text{int}}(v_2, w_2)$ which satisfy the above property.

Proposition (Willems '72)

Let G_1 and G_2 be strictly dissipative with respect to $s_1(v_1, d_1, w_1, z_1)$ and $s_2(v_2, d_2, w_2, z_2)$, where both have an additive structure. Let $V_1(x_1)$ and $V_2(x_2)$ be some storage functions. Suppose $s_{1,\text{int}}(v_1, w_1)$ and $s_{2,\text{int}}(v_2, w_2)$ are interconnection neutral supply functions. Then the system G is strictly dissipative with respect to the supply $s_{\text{ext}}(d_1, d_2, z_1, z_2) := s_{1,\text{ext}}(d_1, z_1) + s_{2,\text{ext}}(d_2, z_2)$ with a storage function $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$.

The function V from the proposition is an example of a structured Lyapunov function.

Stability of two interconnected systems



Proposition (Willems '72)

Let G_1 and G_2 be strictly dissipative with respect to $s_{1,\text{int}}(v_1, w_1)$ and $s_{2,\text{int}}(v_2, w_2)$ with some corresponding storage functions $V_1(x_1)$ and $V_2(x_2)$, respectively. Suppose $V_1(\cdot)$ and $V_2(\cdot)$ are positive definite functions and that $s_{1,\text{int}}(v_1, w_1)$ and $s_{2,\text{int}}(v_2, w_2)$ are interconnection neutral supply rates. Then the system G is stable.

Extensions of these two propositions to a larger number of interconnected systems is straightforward.

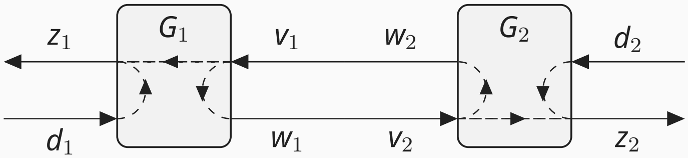
Structured Lyapunov functions (the result)

Two interconnected systems

Two interconnected systems are given by

$$G_i: \quad \begin{pmatrix} \dot{x}_i \\ w_i \\ z_i \end{pmatrix} = \begin{pmatrix} A_i & B_i & E_i \\ C_i & D_i & 0 \\ F_i & K_i & L_i \end{pmatrix} \begin{pmatrix} x_i \\ v_i \\ d_i \end{pmatrix}, \quad i = 1, 2.$$

Hence there is no direct feed-through path that goes from an external input of G_i to an external output of G_j, $i \neq j$.



Two interconnected systems: a theorem

Theorem

Let C_1 and C_2 be full row rank matrices.[1]
Suppose that the system G is strictly dissipative with respect to a quadratic supply function $s_{ext}(d_1, d_2, z_1, z_2)$, which has an additive structure

$$s_{\text{ext}}(d_1, d_2, z_1, z_2) = s_{1,\text{ext}}(d_1, z_1) + s_{2,\text{ext}}(d_2, z_2),$$

and suppose there exists a storage function $V(x_1, x_2)$ of the form

$$V(x_1, x_2) = V_1(x_1) + V_2(x_2),$$

with V_1 and V_2 quadratic functions.

[1] Technical assumption.

Theorem (Continuation)

Then there exist quadratic interconnection neutral supply functions $s_{1,\text{int}}(v_1, w_1)$ and $s_{2,\text{int}}(v_2, w_2)$ such that

$$s_{1,\text{int}}(v_1, w_1) + s_{2,\text{int}}(v_2, w_2) = 0 \quad \text{for } v_1 = w_2, v_2 = w_1,$$

and

i) G_1 is strictly dissipative with respect to the supply $s_1(d_1, v_1, z_1, w_1) := s_{1,\text{ext}}(d_1, z_1) + s_{1,\text{int}}(v_1, w_1)$ with the storage function $V_1(x_1)$,

ii) G_2 is strictly dissipative with respect to the supply $s_2(d_2, v_2, z_2, w_2) := s_{2,\text{ext}}(d_2, z_2) + s_{2,\text{int}}(v_2, w_2)$ with the storage function $V_2(x_2)$.

What does the theorem says?

The theorem is a converse result to the Willems' first proposition.
Together they state that the existence of neutral supply functions such that

$$V_1(x_1) > 0,\ \dot{V}_1(x_1) < s_1(v_1, w_1) \text{ for all } x_1, v_1, w_1 \neq 0,$$
$$V_2(x_2) > 0,\ \dot{V}_2(x_2) < s_2(v_2, w_2) \text{ for all } x_2, v_2, w_2 \neq 0,$$
$$s_1(v_1, w_1) + s_2(v_2, w_2) = 0 \text{ for } v_1 = w_2, v_2 = w_1$$

is equivalent to the existence of additive Lyapunov function

$$V(x) = V_1(x_1) + V_2(x_2) > 0, \dot{V}(x) < 0.$$

There is an analogous result regarding dissipativity with respect to external supply functions.

Acyclic dynamical networks

If the underlying graph Γ is acyclic then the previous theorem can be extended.

There is an additive Lyapunov function

$$V(x) = V_1(x_1) + \cdots + V_m(x_m)$$

$$\Updownarrow$$

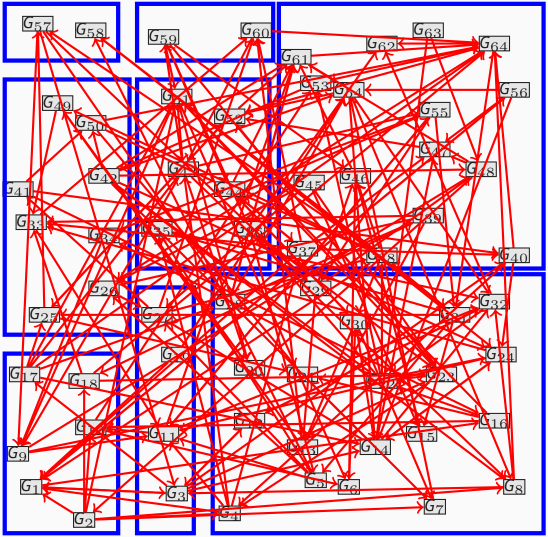
$$\dot{V}_i(x_i) < \sum s_{ij}(v^{ij}, w^{ij})$$

along trajectories x^i, v^{ij}, w^{ij} satisfying

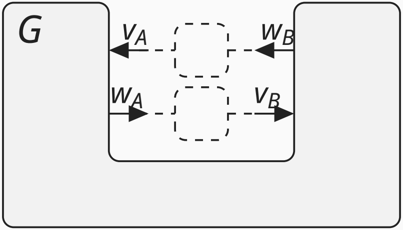
$$s_ij(v^{ij}, w^{ij}) + s_{ji}(v^{ji}, w^{ji}) = 0$$

for all (i,j) such that $(G^i, G^j) \in \Gamma$.

And if the graph is not acyclic?



A corollary: robustness



Proposition

Suppose that some interconnection link in a system is characterized with an interconnection neutral supply function which satisfies the property $s_A(0, w_A) \leq 0$ for all $w_A \neq 0$, $s_B(0, w_B) \leq 0$ for all $w_B \neq 0$ and with a positive definite storage function. Then the system is stable irrespective of whether the interconnection is present or not.

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Thanks for the attention!