Cost of the Null Controllability of Parabolic Partial Differential Equations

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Motivating questions

How large is the control cost of the null controllability?

How does it scale with parameters of the system?

Can we recover physically reasonable relations in the control cost? E.g. "space parameter"^{2} $\sim$ "time" for control of the heat equation $\dot{w} = \Delta w$.

Outline

What is null controllability and its cost?

Null controllability with explicit cost.

Scalable unique continuation principle for $A = -\Delta + V$.

Putting things together.

Setting

X, U Hilbert spaces. A self-adjoint in X, $A \geq \kappa$, $\kappa \in \mathbb{R}$.
$B : U \to X_{-1}$ bounded.

The equation in the unknown $w : [0, T] \to X$

$$\dot{w} = Aw + Bu, \quad w(0) = w_0.$$

u is called the control function, the solution w is called the state.
This equation models a control system.
The solution is given by

$$w(t) = \mathrm{e}^{tA} w_0 + \int_0^t \mathrm{e}^{(t-s)A} Bu(s)\, \mathrm{d}s, \text{ for all } 0 \leq t \leq T.$$

Null controllability

Definition

The control system is null controllable in time T if for any w_0 there exists u such that the state satisfies $w(T) = 0$.

Theorem

The system is null controllable in time T if and only if there exists a constant $c > 0$ such that

$$\int_0^T \|B^* e^{tA^*} \varphi_0\|^2 \, dt \geq c \|e^{TA^*} \varphi_0\|, \text{ for all } \varphi_0.$$

This is called an observability inequality.

Condition for null controllability

$$\dot{w} + Aw = Bu, \quad w(0) \in X, \quad A \geq \kappa.$$

Theorem (Sufficient condition for null controllability, NTTV '18)

Let $d_0 \geq 1$, $d_1 > 0$, $0 < \gamma < 1$ such that for all $\phi \in X$, $\lambda \geq \kappa$:

$$\left\| P_A(-\infty, \lambda)\phi \right\|^2 \leq d_0 e^{d_1 (\lambda - \kappa)^\gamma} \left\| B^* P_A(-\infty, \lambda)\phi \right\|^2.$$

Such inequalities are called spectral inequalities.
Then, for all $T > 0$, $w(0) \in X$, there is an input function u such that $w(T) = 0$, and

$$\int_0^T \|u(t)\|^2 \mathrm{d}T \leq C_{\mathrm{control}} \|w(0)\|^2,$$

AND we know a lot about $C_{\mathrm{control}} = C_{\mathrm{control}}(T, d_0, d_1, B, \kappa)$.

What's new?

We simultaneously treat:

- A with not necessarily discrete spectrum (e.g. full space Laplacian),
- A not necessarily positive (e.g. $A = -\Delta + V$ Schrödinger operator),
- all $T > 0$,
- possibly unbounded input operator B as a mapping from U to X (e.g. for boundary control).

Every single item on this list has somewhere already been treated, but the combination (to our knowledge) not.

What's new?

Improvements:

- explicit dependence of C_{control} on T, d_0, d_1, $\|B\|$, κ,
- optimal at large and small T,
- d_1 dependence is better than in previously known results.

Sharp lower bounds

Theorem (NTTV '18)

For every $T > 0$, if a system is null controllable in time T, the control cost C_{control} satisfies

$$C_{\text{control}} \geq \|B\|^{-2} \begin{cases} \frac{2\kappa(1+\kappa^2)^\beta}{1-\exp(-2\kappa T)} & \text{if } \kappa > 0, \\ T^{-1} & \text{if } \kappa = 0, \\ \frac{-2\kappa(1+\kappa^2)^\beta}{\exp(-2\kappa T)-1} & \text{if } \kappa < 0. \end{cases}$$

These bounds are sharp.

How does C_{control} look like? (1)

Let us first give some clues.

If $T \searrow 0$:

$C_{\text{control}} \sim \exp\left(\frac{C}{T}\right)$. Asymptotically sharp.

If $T \nearrow \infty$:

Recall $A \geq \kappa$. The free system satisfies $w(t) = e^{At}w(0)$.

If $\kappa < 0$: $C_{\text{control}} \sim e^{C\kappa}$. "Large times are your friend." Asymptotically sharp.

If $\kappa = 0$: $C_{\text{control}} \sim T^{-1}$. "Large times are still your friend." Asymptotically sharp.

If $\kappa > 0$: $C_{\text{control}} \geq C > 0$. "Large times don't really help you."

How does C_{control} look like? (2)

Dependence on d_0, d_1:

Recall the assumption was:

$$\left\|P_A(-\infty,\lambda)\phi\right\|^2 \leq d_0 e^{d_1(\lambda-\kappa)^\gamma}\left\|B^* P_A(-\infty,\lambda)\phi\right\|^2, \quad d_0 \geq 1, d_1 > 0.$$

$C_{\text{control}} \sim \exp(C d_1^{1/\gamma})$. Sharp.

$C_{\text{control}} \sim d_0 \exp(C \log(d_0^{1/\gamma}))$. Subexponential in d_0.

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Here it is:

$$C_{\text{control}} = C\frac{d_0}{T} \exp\left(C\left(\frac{d_1}{T^\gamma}\right)^{\frac{1}{1-\gamma}} + \frac{C \ln^{1/\gamma}(d_0(\|B\|^2+1))}{\min\{T^{\frac{\gamma}{1-\gamma}};1\}} - 2\kappa T \right)$$

So far ...

... we have seen that it is nice to have an inequality

$$\left\|P_A(-\infty,\lambda)\phi\right\|^2 \leq d_0 e^{d_1(\lambda-\kappa)^{1/\gamma}} \left\|B^* P_A(-\infty,\lambda)\phi\right\|_Y^2$$

and to know as much as possible about d_0 and d_1.

Scalable quantitative unique continuation principle

Theorem (NTTV '18)

Let $G > 0$, $\delta < G/2$ and $S_{\delta,G}$ a union of δ-balls in $\mathbb{R}^d$ such that every elementary cell of $G \cdot \mathbb{Z}^d$ contains exactly one δ-ball.

Let $V \in L^\infty(\mathbb{R}^d)$.

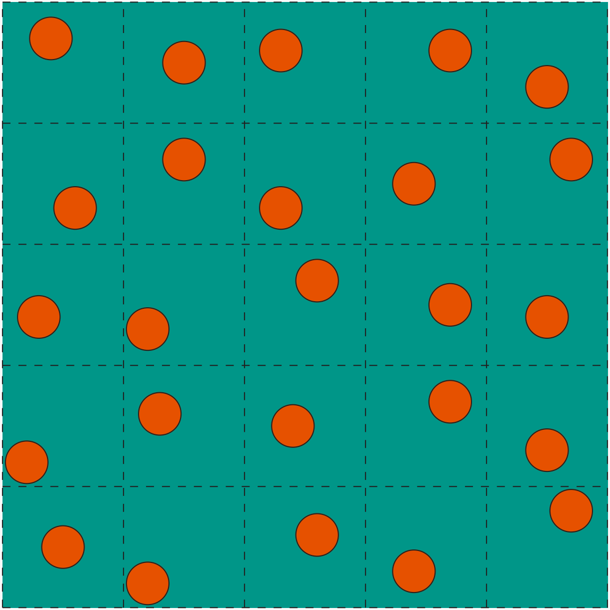
Then, for every $\phi \in L^2(\mathbb{R}^d)$, $\lambda \geq \kappa := \inf \sigma(-\Delta + V)$, we have

$$\|P_{-\Delta+V}(-\infty, \lambda)\phi\|^2_{L^2(\mathbb{R}^d)} \leq d_0 e^{d_1\sqrt{\lambda-\kappa}} \|\chi_{S_{\delta,G}} \cdot P_{-\Delta+V}(-\infty, \lambda)\phi\|^2_{L^2(\mathbb{R}^d)},$$

where

$$d_0 = (\delta/G)^{C(1+G^{4/3}\|V-\kappa\|_\infty^{2/3})}, \quad d_1 = C\ln(\delta/G)G.$$

Geometric setting



Some remarks

Our system is a heat system with a heat generation term V, and where the control function acts on the set $S_{\delta,G}$:

$$\dot{w} = (-\Delta + V)w + \chi_{S_{\delta,G}} u.$$

- The theorem generalizes in a natural way from $\mathbb{R}^d$ to generalized rectangles (finite cubes, half-spaces, slabs in $\mathbb{R}^d$, ...).
- Instead of Δ we can have a 2nd order elliptic differential operator.
- Proof relies on Carleman estimates.

Putting things together

Let $V \in L^\infty(\mathbb{R}^d)$, G, δ, $S_{\delta,G}$ as before, and $T > 0$. Then

$$\partial_t \phi(x,t) - \Delta\phi(x,t) + V(x)\phi(x,t) = \chi_{S_{G,\delta}}(x)u(x,t) \quad \text{in } L^2(\mathbb{R}^d \times [0,T])$$

is null controllable with cost

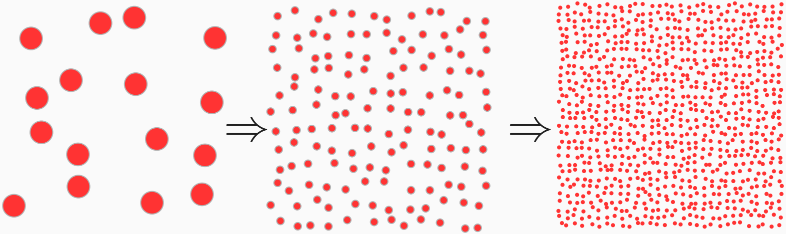
$$C_{\text{control}} = C\frac{d_0}{T}\exp\left(\frac{Cd_1^2}{T} + \frac{C\ln^2(d_0)}{\min\{T^2;1\}} - 2\|V\|_\infty T\right)$$

where

$$d_0 = \left(\frac{\delta}{G}\right)^{C(1+G^{4/3}\|V\|_\infty^{2/3})}, \quad d_1 = C\ln(\delta/G)G.$$

This looks a bit too complicated! So let us have a look at some particular cases.

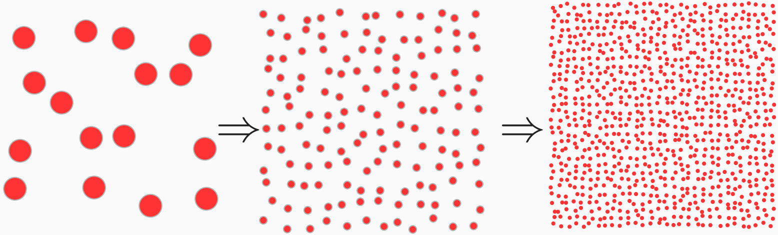
Homogenization



Let $V \geq 0$, and
let us send $G \searrow 0$ while keeping δ/G constant.

$$C_{\text{control}} \to \frac{C}{T}.$$

Homogenization



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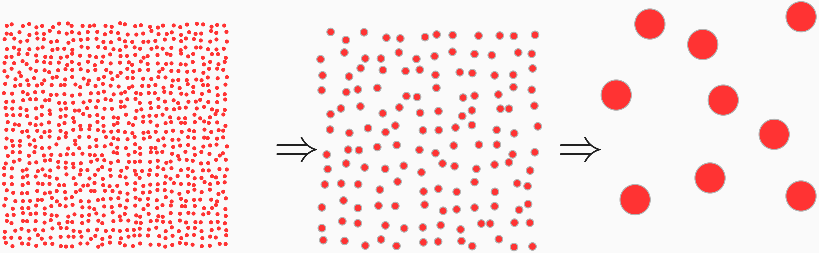
$$C_{\text{control}} \to \frac{C}{T}.$$

We see that homogenization not only annihilates the exponential term which is characteristic for the heat equation, but also the influence of a non-negative potential on the control cost estimate disappears.

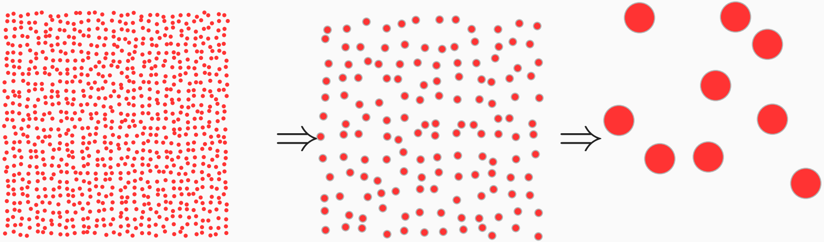
De-homogenization



Let $V = 0$ and
let us send $G \to \infty$ while keeping δ / G constant.

$$C_{\text{control}} = \frac{C}{T} \exp\left(C \frac{G^2}{T} + C \right).$$

De-homogenization



Let $V = 0$ and
let us send $G \to \infty$ while keeping δ/G constant.

$$C_{\text{control}} = \frac{C}{T}\exp\left(C\frac{G^2}{T} + C\right).$$

Diverges when $G \nearrow \infty$. This can be accommodated by giving more time, i.e. $T \sim G^2$. Compare $\partial_t \sim \Delta$ in heat equation to $T \sim G^2$ in control cost. Same relation as orders of time and space derivatives in the PDE. ☺

Fractional heat equation

Let $\theta \in (1/2, \infty)$.

$$\frac{\partial}{\partial t} u + (-\Delta)^\theta u = \chi_{S_{\delta,G} \cap \mathbb{R}^d} f, \quad u, f \in L^2([0,T] \times \mathbb{R}^d),$$
$$u(0,\cdot) = u_0 \in L^2(\mathbb{R}^d),$$

It is known in dimension $d = 1$ that the fractional heat equation on bounded domains is null controllable if and only if $\theta > 1/2$.
We can show that the fractional heat equation is null controllable for $\theta > 1/2$ in $\mathbb{R}^d$ for any dimension.
The spectral inequality for the fractional Laplacian follows from the spectral inequality for the ordinary Laplacian.
The bound on C_{control} looks the same as for the ordinary heat equation but with $\gamma = \frac{1}{2\theta}$.

Thanks for the attention!