Mixed control of vibrational systems

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Setting

Linear vibrational systems, modeled as a 2nd order matrix ODE

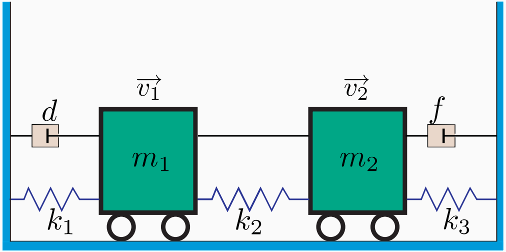
$$M\ddot{q} + D\dot{q} + Kq = F$$

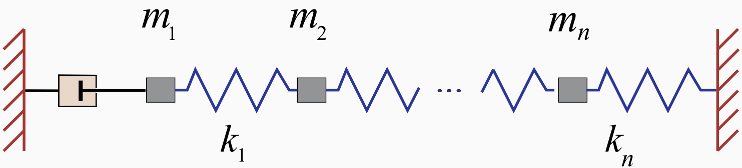
M mass matrix ($M > 0$)

D damping matrix ($D \geq 0$)

K stiffness matrix ($K > 0$)

F external force





Aim

Aim: Attenuate unwanted vibrations of the system by the use of passive damping.

In other words, find an appropriate damping matrix D such that the system vibrates as little as possible.

Why?

Optimization criteria

For a random/unknown external force it is typical to use the machinery of the control theory.

Many different optimization criteria from control theory, most common ones:

- ➔ H_2 norm
- ➔ H_∞ norm

H_2 norm criterion: external force modeled by (white/colored) noise, we obtain best damping for a "typical" external force. A good choice for a large class of vibrational systems (non-critical systems, where external environment changes).

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Let

$$y = \begin{bmatrix} C_1 q \\ C_2 \dot{q} \end{bmatrix}$$

be the measured output of the system.

H_2 norm of a vibrational system

$$G = G(D) = \begin{cases} M\ddot{q} + D\dot{q} + Kq = B_2 u, \\ y = \begin{bmatrix} C_1 q \\ C_2 \dot{q} \end{bmatrix}. \end{cases}$$

Let

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ M^{-1}B_2 \end{bmatrix}, \; A = A(D) = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}.$$

Then the H_2 norm of the system is given by $\mathrm{Tr}(C^* C X)$, where X is the solution of the Lyapunov equation

$$AX + XA^* = -BB^*.$$

Different linearizations (transformations to 1st order ODE) of the vibrational system amount to different state transformations of the system (A, B, C).

A problem

Proposition

The optimization problem

$$\min_{\text{feasible } D} \|G(D)\|_2$$

is not well posed.

A problem

ConDyS

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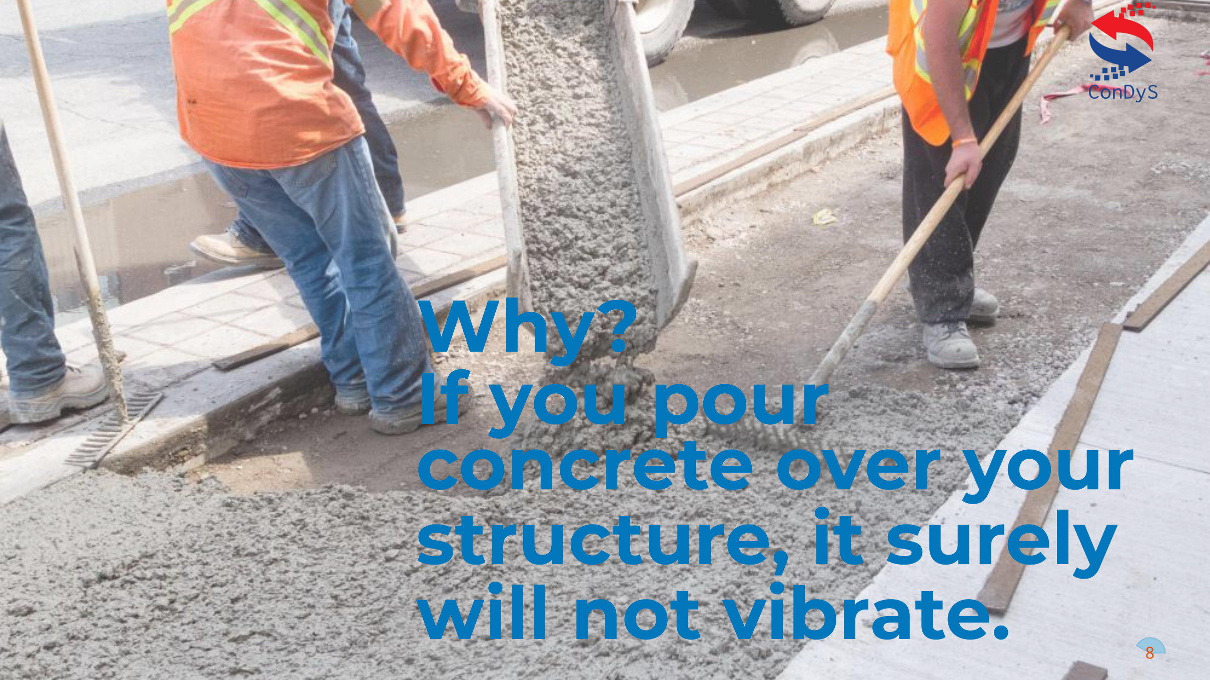
is not well posed.

What is the best damping matrix for the H_2 norm criterion?

$$D = \infty.$$

Also when doing numerics, for some configurations one obtains that the damping coefficient (viscosity of the damper) should be as large as possible.



ConDyS

Why?
If you pour concrete over your structure, it surely will not vibrate.

Why?

H_2 norm can be interpreted as a measure of the average output energy over the impulsive inputs.

But because we calculate H_2 norm of the linearized system, half the impulsive inputs are not taken into account.

This seems to be a general issue when the (first order) control system is obtain by a linearization from the higher order systems.

H_2 norm of a homogeneous system

We define a H_2 norm of the homogeneous system which takes into account the initial conditions.

It is defined by

$$\int_{\|q_0\|^2+\|\dot{q}_0\|^2=1} \int_0^\infty e(t; q_0, \dot{q}_0)\, \mathrm{d}t\, \mathrm{d}\sigma,$$

where e is the energy of the part of the system (or whatever else C measures).

This norm can be written as $\mathrm{Tr}(C^* C Y)$, where Y solves $AY + YA^* = -Z_\sigma$.

Mixed H_2 norm

The issue with the H_2 norm of the corresponding homogeneous system is that it does not carry any information about the external forces.

The issue with the standard H_2 norm is that it does not carry all the needed information about the initial data.
A natural choice is to try to combine these two norms by taking their convex sum.

Let $0 \le p \le 1$. Then p–mixed H_2 norm of the system G is given by

$$\mathrm{Tr}(\tilde{C}^*\tilde{C}X), \text{ where } \tilde{A}X + X\tilde{A}^* = -p\tilde{Z}_\sigma - (1-p)\tilde{B}\tilde{B}^*,$$

where $(\tilde{A}, \tilde{B}, \tilde{C})$ is a linearization of the system G. p–mixed H_2 norm does not depend on the choice of the linearization.

One can also think of it as the standard H_2 norm with an additional constraint taking into account the initial data.

A convenient linearization

$$\breve{A} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\breve{D} \end{bmatrix}, \quad \breve{C} = \begin{bmatrix} \breve{C}_1 & 0 \\ 0 & \breve{C}_2 \end{bmatrix}, \quad \breve{B} = \begin{bmatrix} 0 \\ \breve{B}_2 \end{bmatrix},$$

where $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n)$ are square roots of the eigen–frequencies of the corresponding undamped system ($D = 0$).

A natural choice for C_1, C_2, B_2 and σ gives:
$\breve{Z}_\sigma = \frac{1}{2n} Z$, $Z = \mathrm{diag}(Z_1, Z_1)$, $Z_1 = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$,
$\breve{B}_2 = Z_1,$
$\breve{C}^* \breve{C} = \frac{1}{2} Z$.
Hence, p-mixed H_2 norm is then given by

$$Tr(ZX), \text{ where } \breve{A}X + X\breve{A}^* = -\begin{bmatrix} pZ_1 & 0 \\ 0 & Z_1 \end{bmatrix}.$$

Global optimization problem revisited

ConDyS

Theorem

Let $Z_1 = I$. *Let*

$$\mathcal{D}_s = \{\breve{D} \colon \breve{D} \geq 0 \text{ and the corresponding } \breve{A} \text{ is stable}\}.$$

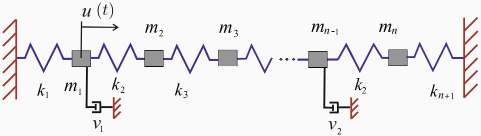
Then for all $0 < p \leq 1$ *there exists a unique global minimum of the following optimization problem:*

$$\text{minimize } \operatorname{Tr}(X) \text{ subject to } \breve{A}X + X\breve{A}^* = -\begin{bmatrix} pI & 0 \\ 0 & I \end{bmatrix} \text{ and } \breve{D} \in \mathcal{D}_s.$$

The minimum is attained at $\breve{D} = \sqrt{\frac{2(1+p)}{p}}\Omega$. ima smisla čak i za p=infty, što to znači?

For $p = 0$ we get ∞, the case of the standard H_2 norm.
For $p = 1$ we get 2Ω, which was already known as the global optimal matrix for this criterion.

Numerical experiments - setting



We assume internal damping of the form $\alpha \cdot 2\Omega$.
We take

$$n = 100; \quad \alpha = 0.02$$

$$k_i = 100, \quad \forall i; \qquad m_i = \begin{cases} 200 - 2i, & i = 1, \dots, 50, \\ i + 50, & i = 51, \dots, 100. \end{cases}$$

Primary excitation matrix B_2 is applied to 5 consecutive masses, i.e.

$$B_2(1:5, 1:5) = \text{diag}(5, 4, 3, 2, 1).$$

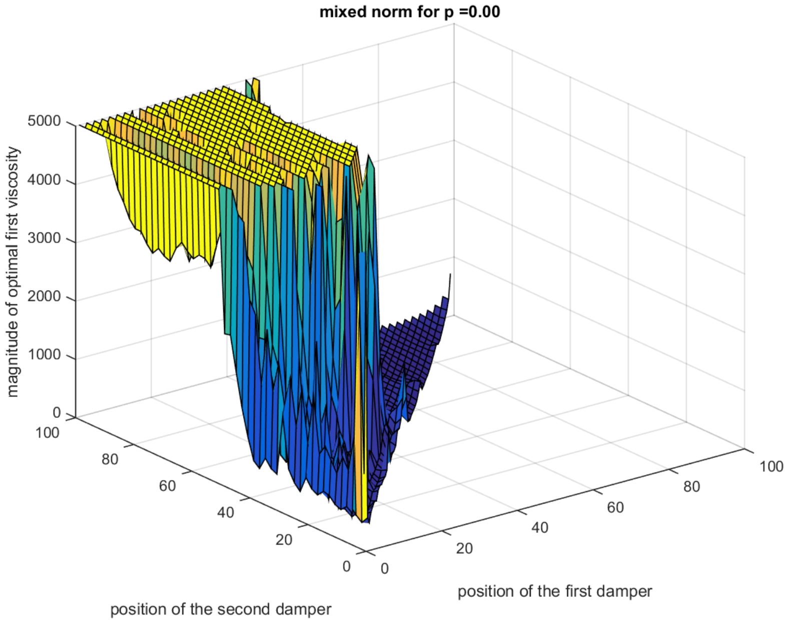
Numerical experiments - setting

We are interested in the 10 states equally distributed, only displacements ($C_2 = 0$)

$$C_1(1:10, 46:55) = I_{10\times 10}$$

The geometry of the external damping is determined by two dampers:

$$D_{\text{ext}} = \begin{bmatrix} e_i & e_j \end{bmatrix} \operatorname{diag}(v_1, v_2) \begin{bmatrix} e_i^T \\ e_j^T \end{bmatrix}$$


mixed norm for p =0.00
magnitude of optimal first viscosity
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2000
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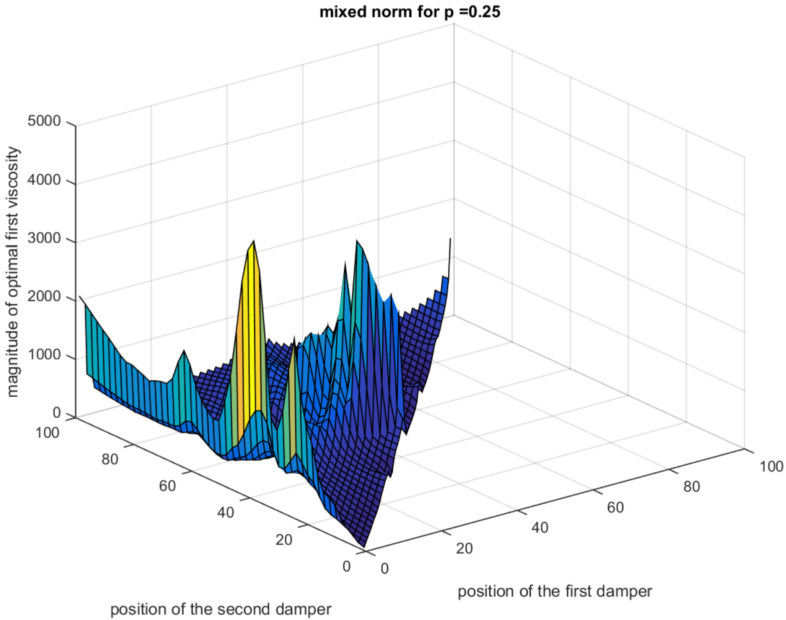
position of the second damper
position of the first damper

ConDyS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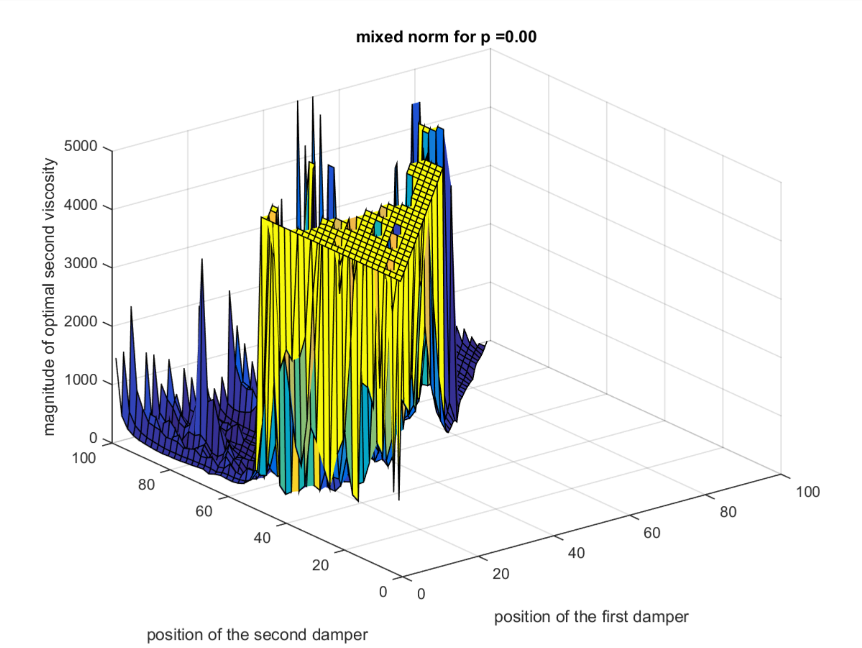
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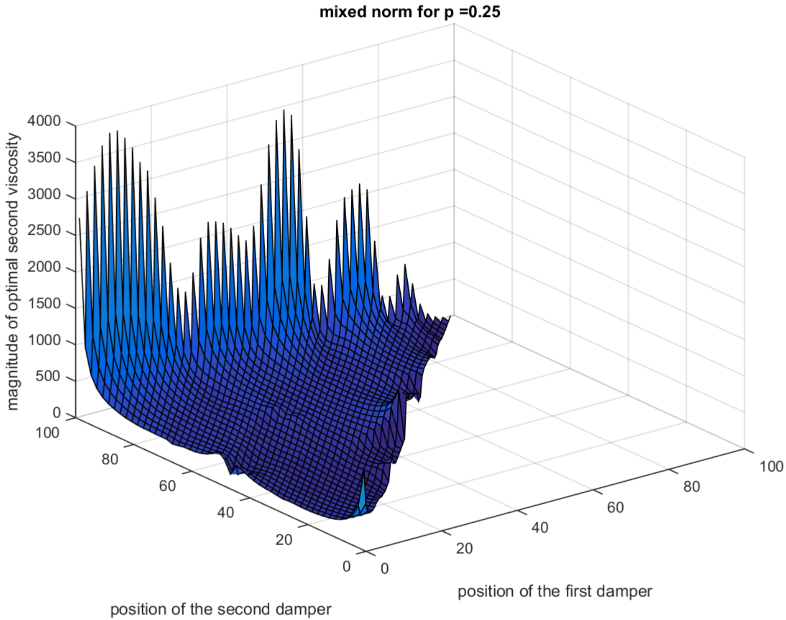
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Thanks for the attention!