Optimal Passive Control Of Vibrational Systems Using Mixed Performance Measures

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Setting

Linear vibrational system, modeled as a 2nd order matrix differential equation

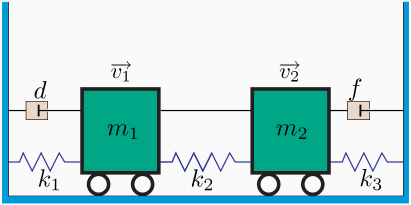
$$M\ddot{q} + D\dot{q} + Kq = F$$

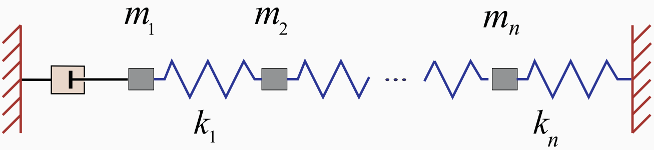
M mass matrix

D damping matrix

K stiffness matrix

F external force





Aim

System will have N modes of vibration, where N is the dimension of the system. Not all modes are dangerous, usually there is a range of dangerous ones.

Aim: Attenuate unwanted vibrations of the system by the use of passive damping.

In other words, find an appropriate damping matrix D such that the system vibrates as little as possible.

Why?

Why?

How to do it?

Important classes of approaches:

➔ based on the analysis of stationary system (external force $F = 0$, excitation by the initial condition), some interesting ones:

- $\rightarrow$ based on eigenvalues (e.g. max $\Re\lambda$, max $\frac{\Re\lambda}{|\lambda|}$)
- $\rightarrow$ based on the total energy (e.g. max $\int_0^\infty E(t)\mathrm{d}t$, avg. $\int_0^\infty E(t)\mathrm{d}t$)

➔ based on the analysis of excitation by a particular external force

- $\rightarrow$ harmonic excitation
- $\rightarrow$ periodic non-harmonic excitation

How to do it?

For a random/unknown external force we can use the machinery of the control theory.
Again different criteria, most common ones:

- ➔ H_2 norm
- ➔ H_∞ norm

H_2 norm criterion: external force modeled by (white/coloured) noise, we obtain best damping for a "typical" external force.

H_2 norm criterion is a good choice for a large class of vibrational systems (non-critical systems, where external environment changes).

H_2 norm of a vibrational system

$$G = G(D) = \begin{cases} M\ddot{q} + D\dot{q} + Kq = B_2 u, \\ y = \begin{bmatrix} C_1 q \\ C_2 \dot{q} \end{bmatrix}. \end{cases}$$

Let

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ M^{-1}B_2 \end{bmatrix}, \; A = A(D) = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}.$$

Then the H_2 norm of the system is given by $\mathrm{Tr}(C^* C X)$, where X is the solution of the Lyapunov equation

$$AX + XA^* = -BB^*.$$

Different linearizations of the vibrational system amount to different state transformations of the system (A, B, C).

A problem

Proposition

The optimization problem

$$\min_{\text{feasible } D} \|G(D)\|_2$$

is not well posed.

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The optimization problem

$$\min_{\text{feasible } D} \|G(D)\|_2$$

is not well posed.

What is the best damping matrix for the H_2 norm criterion?

$$D = \infty.$$

Also when doing numerics, for some configurations one obtains that the damping coefficient should be as large as possible.

Why?
If you pour concrete over your structure, it surely will not vibrate.

Why?

H_2 norm can be interpreted as a measure of the average output energy over the impulsive inputs.

But because we calculate H_2 norm of the linearized system, half the impulsive inputs are not taken into account.

This seems to be a general issue when the (first order) control system is obtain by a linearization from the higher order systems.

H_2 norm of a homogeneous system

We generalize the total energy approach for the measurement of unwanted vibrations of a homogeneous vibrational system which is in a way counterpart to the H_2 norm of the system.
We take $u = 0$ but include the initial conditions $q(0) = q_0$, $\dot{q}(0) = \dot{q}_0$.

The H_2 norm of the homogeneous system is defined by

$$\int_{\|q_0\|^2+\|\dot{q}_0\|^2=1} \int_0^\infty e(t; q_0, \dot{q}_0)\, \mathrm{d}t\, \mathrm{d}\sigma,$$

where e is the energy of the part of the system (or something else C measures) and σ is a surface measure on the unit sphere.

This norm can be written as $\mathrm{Tr}(C^* C Y)$, where Y solves $AY + YA^* = -Z_\sigma$, Z_σ depending only on the measure σ.

Mixed H_2 norm

The issue with the H_2 norm of the corresponding homogeneous system is that it does not carry any information about the external forces, and the issue with the standard H_2 norm is that it does not carry all the needed information about the initial data. A natural choice is to try to combine these two norms by taking their convex sum.

Let $0 < p < 1$. We define p–mixed H_2 norm of the system G by

$$\mathrm{Tr}(\tilde{C}^* \tilde{C} X), \text{ where } \tilde{A} X + X \tilde{A}^* = -p \tilde{Z}_\sigma - (1-p) \tilde{B} \tilde{B}^*.$$

p–mixed H_2 norm does not depend on the choice of the linearization.

One can also think of it as the standard H_2 norm with an additional constraint taking into account the initial data.

A convenient linearization

$$\breve{A} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\breve{D} \end{bmatrix}, \quad \breve{C} = \begin{bmatrix} \breve{C}_1 & 0 \\ 0 & \breve{C}_2 \end{bmatrix}, \quad \breve{B} = \begin{bmatrix} 0 \\ \breve{B}_2 \end{bmatrix},$$

where $\Omega = \text{diag}(\omega_1, \ldots, \omega_n)$ are square roots of the eigen–frequencies of the corresponding undamped system (with $D = 0$).

A natural choice for C_1, C_2, B_2 and σ gives:
$\breve{Z}_\sigma = \frac{1}{2n}Z$, $Z = \text{diag}(Z_1, Z_1)$, $Z_1 = \text{diag}(1, \ldots, 1, 0, \ldots, 0)$,
$\breve{B}_2 = Z_1,$
$\breve{C}^*\breve{C} = \frac{1}{2}Z$.
Hence, p-mixed H_2 norm is then given by

$$Tr(ZX), \text{ where } \breve{A}X + X\breve{A}^* = -\begin{bmatrix} pZ_1 & 0 \\ 0 & Z_1 \end{bmatrix}.$$

Global optimization problem revisited

Theorem

Let $Z_1 = I$. *Let*

$$\mathcal{D}_s = \{\breve{D} \in \mathbb{R}^{n\times n} : \breve{D} \geq 0 \text{ and the corresponding } \breve{A} \text{ is stable}\}.$$

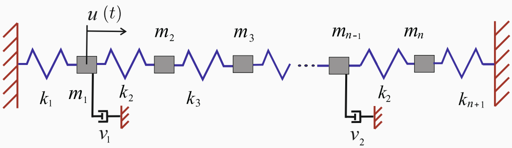
Then for all $0 < p < 1$ *there exists a unique global minimum of the following optimization problem:*

$$\text{minimize } \operatorname{Tr}(X) \text{ subject to } \breve{A}X + X\breve{A}^* = -\begin{bmatrix} pI & 0 \\ 0 & I \end{bmatrix} \text{ and } \breve{D} \in \mathcal{D}_s.$$

The minimum is attained at $\breve{D} = \sqrt{\frac{2(1+p)}{p}}\Omega$.

For $p = 0$ we get ∞, the case of the standard H_2 norm.
For $p = 1$ we get 2Ω, which was already known as the global optimal matrix for this criterion.

Numerical experiments - setting



We assume internal damping of the form $\alpha \cdot 2\Omega$.
We take

$$n = 100; \quad \alpha = 0.02$$

$$k_i = 100, \quad \forall i; \qquad m_i = \begin{cases} 200 - 2i, & i = 1, \ldots, 50, \\ i + 50, & i = 51, \ldots, 100. \end{cases}$$

Primary excitation matrix B_2 is applied to 5 consecutive masses, i.e.

$$B_2(1:5, 1:5) = \text{diag}(5, 4, 3, 2, 1).$$

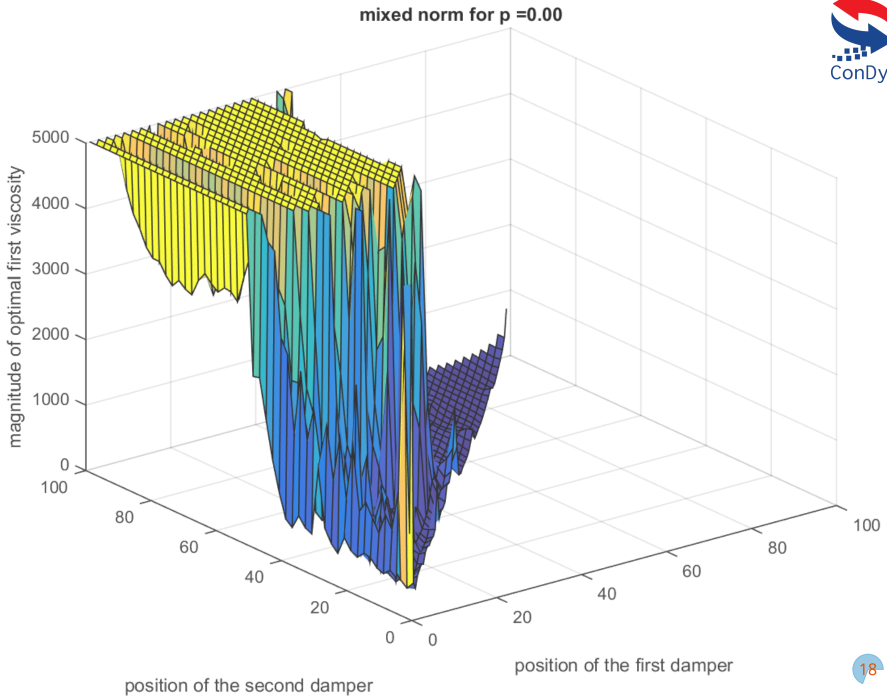
Numerical experiments - setting

We are interested in the 10 states equally distributed, only displacements ($C_2 = 0$)

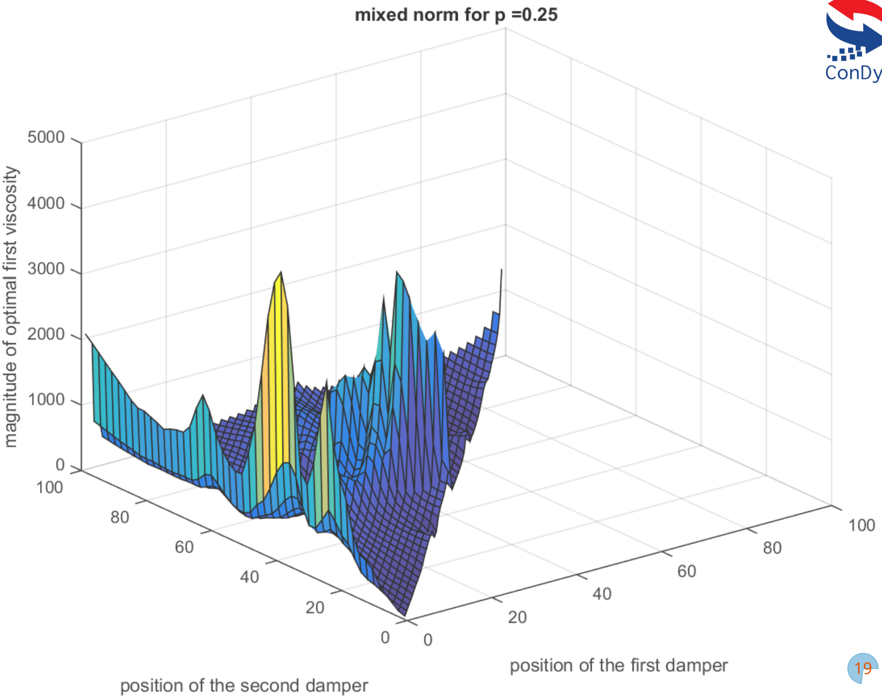
$$C_1(1:10, 46:55) = I_{10\times 10}$$

The geometry of the external damping is determined by two dampers:

$$D_{\text{ext}} = \begin{bmatrix} e_i & e_j \end{bmatrix} \operatorname{diag}(v_1, v_2) \begin{bmatrix} e_i^T \\ e_j^T \end{bmatrix}$$


mixed norm for p =0.00
magnitude of optimal first viscosity
5000
4000
3000
2000
1000
0
100
80
60
40
20
0
0
20
40
60
80
100
position of the second damper
position of the first damper

ConDyS


mixed norm for p =0.25
magnitude of optimal first viscosity
5000
4000
3000
2000
1000
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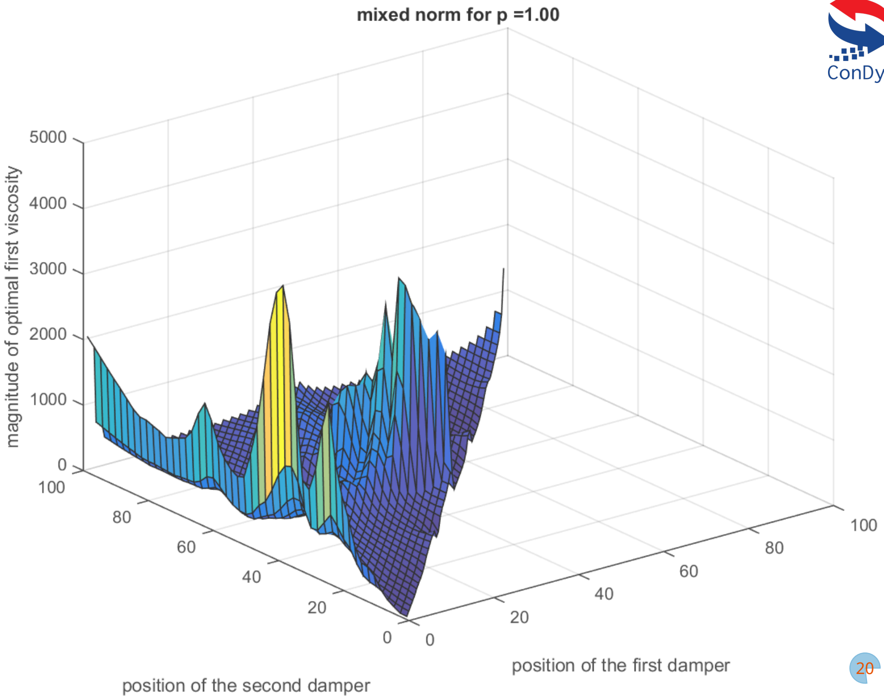
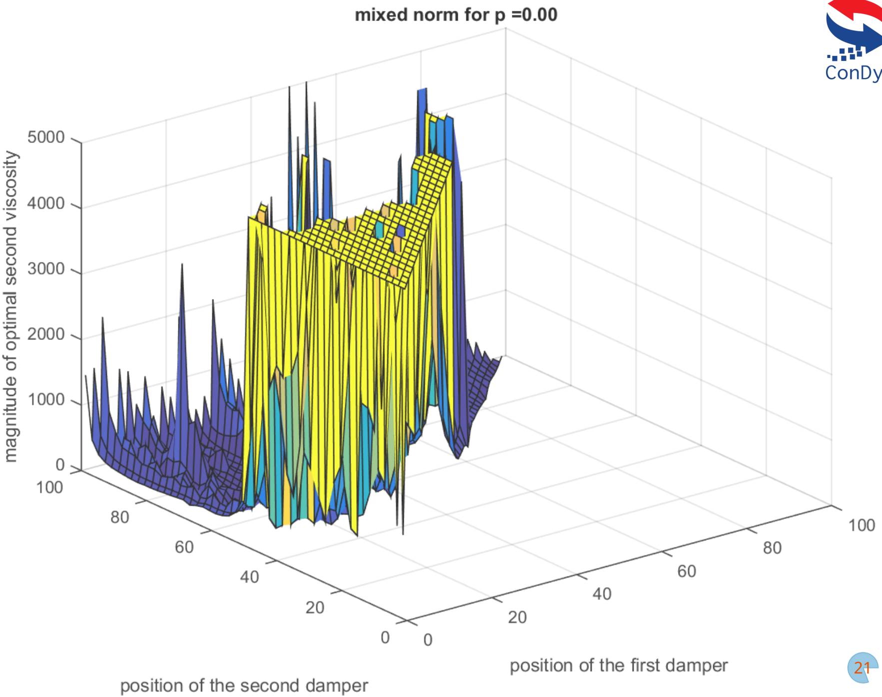
100
80
60
40
20
0
position of the second damper
0
20
40
60
80
100
position of the first damper

ConDyS


mixed norm for p =1.00
magnitude of optimal first viscosity
0
1000
2000
3000
4000
5000
position of the second damper
0
20
40
60
80
100
position of the first damper
0
20
40
60
80
100

ConDyS


mixed norm for p =0.00
magnitude of optimal second viscosity
5000
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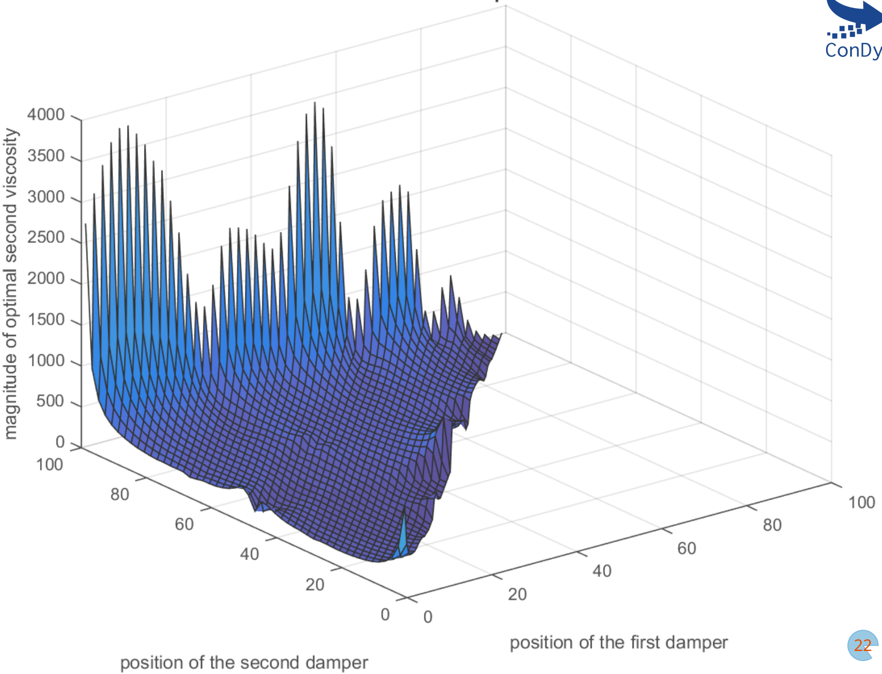
position of the second damper
position of the first damper

mixed norm for p =0.25



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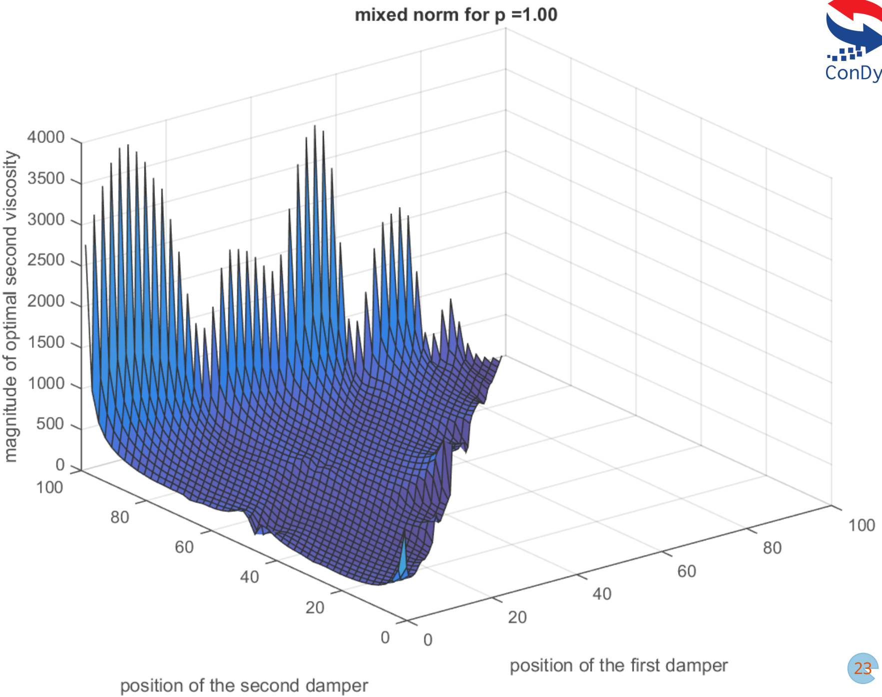
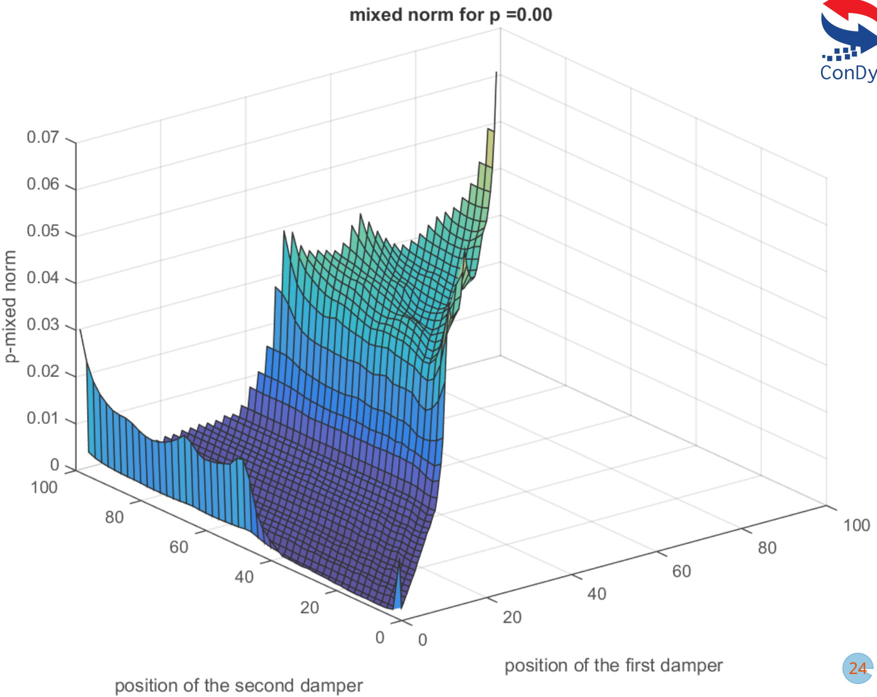
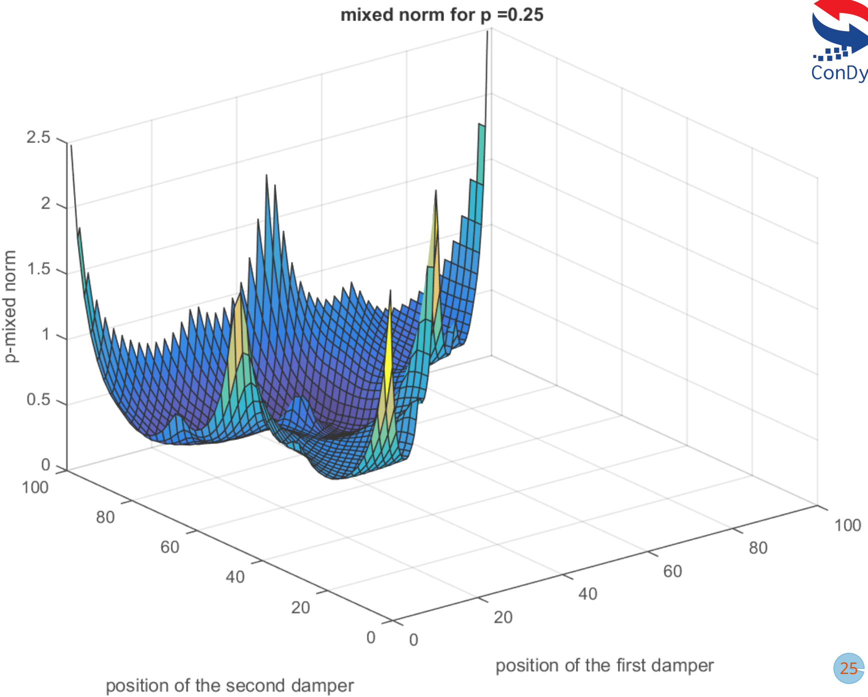

mixed norm for p =1.00
magnitude of optimal second viscosity
4000
3500
3000
2500
2000
1500
1000
500
0
100
80
60
40
20
0
position of the second damper
0
20
40
60
80
100
position of the first damper

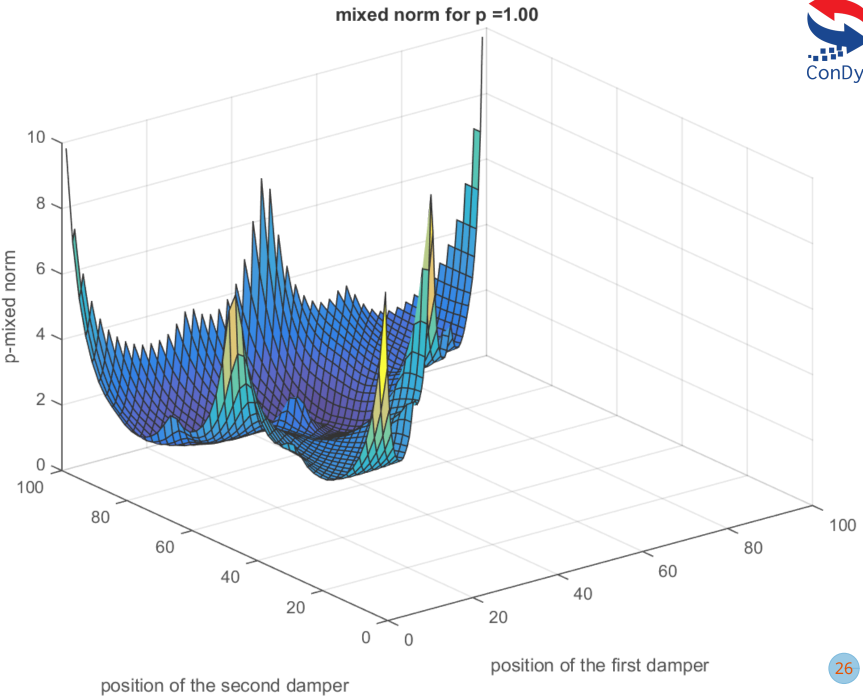
ConDyS


mixed norm for p =0.00
p-mixed norm
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
100
80
60
40
20
0
0
20
40
60
80
100
position of the first damper
position of the second damper

ConDyS


mixed norm for p =0.25
p-mixed norm
2.5
2
1.5
1
0.5
0
100
80
60
40
20
0
0
20
40
60
80
100
position of the second damper
position of the first damper

ConDyS


mixed norm for p =1.00
p-mixed norm
0
2
4
6
8
10
position of the second damper
0
20
40
60
80
100
position of the first damper
0
20
40
60
80
100

Thanks for the attention!