Greedy Algorithm for Parameter Dependent Operator Lyapunov Equations. Application to Control Problems

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VIII PDEs, optimal design and numerics
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We consider a family of parameter-dependent operator Lyapunov equations

$$A_\nu P_\nu + P_\nu A_\nu^* = -Q_\nu \qquad (OLE_\nu)$$

- ν – a parameter ranging over compact set $\mathcal{N} \subseteq \mathbf{R}^d$
- A_ν – an unbounded operator on a Hilbert space X
- Q_ν – a bounded operator on X, $Q_\nu \geq 0$
- P_ν – the solution

Problem

Find the efficient algorithm for solving (OLE_ν) for a wide range of parameters.

Assumptions

For each ν

- $D(A_\nu)$ is dense in X
- the operator A_ν is closed and stable

Then there exists a unique nonnegative solution $P \in \mathcal{L}(X)$

$$P_\nu = \int_0^\infty e^{tA_\nu} Q_\nu e^{tA_\nu^*} dt$$

Different methods for computing the solution.

- Bartels, Stewart *Comm. ACM*, 1972. - the Schur decomposition
- Saad (1990) - Krylov suvspace methods
- Simoncini *SIAM Rev.*, 2016. - iterative methods

Computational expensive.

Can we construct the **solution manifold**

$$\mathcal{P} = \{P_\nu : \nu \in K\}$$

without applying the above methods for each new value of ν?

The idea

To determine a finite number of values of ν that yield the best possible approximation of the solution manifold $\mathcal{P}$

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In order to achieve this goal we rely on greedy algorithms and reduced bases methods for parameter dependent PDEs or abstract equations in Banach spaces.

- A. Cohen, R. DeVore, Kolmogorov widths under holomorphic mappings, *IMA Journal on Numerical Analysis*, 2016
- A. Cohen, R. DeVore, Approximation of high-dimensional parametric PDEs, *Acta Numer.*, 2015.
- Y. Maday, O. Mula, A. T. Patera, M. Yano, The generalized Empirical Interpolation Method..., *Computer Methods in Applied Mechanics and Engineering*, 2015.

The pure greedy method

X – a Banach space $K \subset X$ – a compact subset.

- The method approximates K by a a series of finite dimensional linear spaces V_n (a linear method).
- Offline procedure generates approximation subspace within given precision error; Online routine calculates approximations for any element in K.

The algorithm

The first step Choose $x_1 \in K$ such that

$$\|x_1\|_X = \max_{x \in K} \|x\|_X .$$

The general step Having found $x_1..x_n$, denote $V_n = \mathrm{span}\{x_1, \dots, x_n\}$. Choose the next element

$$x_{n+1} := \arg\max_{x \in K} \mathrm{dist}(x, V_n) . \tag{1}$$

The algorithm stops when $\sigma_n(K) := \max_{x \in K} \mathrm{dist}(x, V_n)$ becomes less than the given tolerance ε.

Efficiency

In order to estimate **the efficiency of the (weak) greedy algorithm** we compare its approximation rates $\sigma_n(K)$ with the best possible one.

The Kolmogorov n width, $d_n(K)$

– measures how well K can be approximated by a subspace in X of a fixed dimension n.

$$d_n(K) := \inf_{\dim Y = n} \sup_{x \in K} \inf_{y \in Y} \|x - y\|_X .$$

Thus $d_n(K)$ represents optimal approximation performance that can be obtained by a n-dimensional linear space.
The greedy approximation rates have same decay as the Kolmogorov widths.

Theorem

(Cohen, DeVore '15) [3]
For any $\alpha > 0, C_0 > 0$

$$d_n(K) \le C_0 n^{-\alpha} \quad \Longrightarrow \quad \sigma_n(K) \le C_1 n^{-\alpha}, \quad k \in \mathbf{N},$$

where $C_1 := C_1(\alpha, C_0, \gamma)$.

[3] A. Cohen, R. DeVore, *Acta Numerica*, 2015.

Performance obstacles

- The set K in general consists of infinitely many vectors.
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Practical realisations depends crucially on an existence of an appropriate surrogate!

Implementation: Residual Analysis

Knowing P_1 how to measure

$$\mathrm{dist}(P_1 - P_\nu)$$

without knowing P_ν?

Check residual

$$R_\nu(P_1) := A_\nu P_1 - P_1 A_\nu + B_\nu B_\nu^*$$

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Theorem

Suppose that

1) A_ν *is sectorial, i.e it is a generator of an analytical semigroup ;*
2) $D(A_{\nu_1}) = D(A_{\nu_2})$ *and* $D(A_{\nu_1}^*) = D(A_{\nu_2}^*)$ *for* $\nu_1,\ \nu_2 \in \mathcal{N}$.

Then

$$||R_\nu|| \sim ||P_1 - P_\nu||$$

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Tricky part - functional setting (norms in which spaces?)

Result in finite dimensional setting

N.T. Son, T. Stykel *Siam J. Matrix Anal. Appl.*, 2017,

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Then

$$||R_\nu||_{\mathcal{L}(X_1^d, X_{-1})} \sim ||P_1 - P_\nu||_{\mathcal{L}(X_1^d, X)}$$

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Collateral result:

Theorem

Lyapunov operator $L_A(P) = AP + PA^*$ *is a bounded and coercive operator from* $\mathcal{L}(X_1^d, X)$ *to* $\mathcal{L}(X_1^d, X_{-1})$.

Control problem

Consider the control system

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + Bu(t), \qquad 0 \leq t \leq T \\ x(0) &= x_0 \end{cases}$$

where B is an admissible control operator.
Suppose that x_T is a reachable state.
Then the optimal norm control $\hat{u}$ is of the type

$$\hat{u} = B^* e^{(T-t)A^*} \phi_T$$

for some vector ϕ_T which corresponds to initial datum of the adjoint equation.
In addition, the following equation holds

$$x_T - e^{tA} x_0 = \Lambda_T \phi_T,$$

where Λ_T is the Gramian operator

$$\Lambda_T = \int_0^T e^{tA} B B^* e^{tA} dt$$

The minimal control energy is given by

$$\|\hat{u}\|^2 = \Lambda_T \phi_T \cdot \phi_T.$$

For dissipative systems Λ_T can be well approximated by the infinite time Gramian operator.

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Solving for Λ_∞ is much easier than constructing Λ_T (which satisfies differential Lyapunov equation).

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But we even want to avoid solving for Λ_∞ !

We introduce parameter dependence

$$\begin{cases} \frac{d}{dt} x_\nu(t) = A_\nu x_\nu(t) + B_\nu u_\nu(t), & 0 \le t \le T \\ x_\nu(0) = x_{0,\nu} \end{cases}$$

We apply the greedy algorithm for solving (approximately) $\Lambda_{\infty,\nu}$

The algorithm is independent of x_0, x_T and T!

Example 1: 1D Heat Equation

$$\begin{cases} \frac{\partial}{\partial t} z - \nu \Delta z = 0 & \text{in} \quad (0,1) \times (0,T), \\ z(0,t) = 0, & \quad z(1,t) = u_\nu(t), \\ z(x,0) = z_0. & \end{cases}$$

The parameter ν ranges within $\mathcal{N} = [0.7, 1300]$

The greedy algorithm has been applied with

- discretized system of dimension $N = 40$,
- $\epsilon = 0.01$,
- uniform discretization of $\mathcal{N}$ in $l = 100$.

The offline algorithm stops **after only one** iteration in approximately 0.06 seconds!

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By change of variables:

$$A_\nu = \nu A \implies \Lambda_{\infty,\nu} = \nu \Lambda_\infty$$

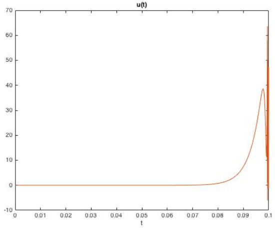
(Holds just for $T = \infty$!)

Example 1: 1D Heat Equation - Online part

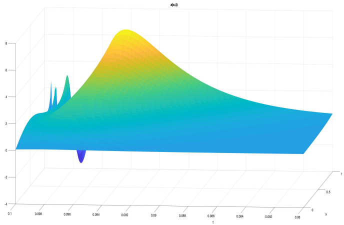
We aim to steer the system

- from $z_0 = 0$ to $z_1 = \sin(\pi x)$
- in time $T = 0.1$
- for $\nu = 23$

Calculation of the approximate Gramian is rather straightforward.
It is applied for construction of the optimal control.
It drives the system to final state z^1 within the error $|z^1 - z(T)| = 3.77 \times 10^{-5}$.



a)



b)

Figure: Evolution of a) the approximate control and b) the solution of semi-discretized example problem.

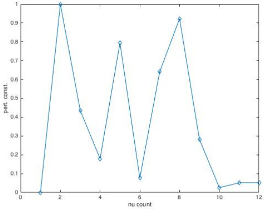
Example 2: Anisotropic 2D Heat Equation

$$\begin{array}{lll} \frac{\partial}{\partial t} z - \Delta_\nu z = 0 & \text{in} & (0,1)^2 \times (0,T), \\ z(x,t) = v_0(x,t), & \text{for} & x \in \partial([0,1]^2) \\ z(x,0) = 0 & & \end{array}$$

- $\Delta_\nu = \frac{\partial^2}{\partial x_1^2} + (1+\nu)\frac{\partial^2}{\partial x_2^2}, \qquad \nu \in \mathcal{N} = [0,1]$
- $$v_0(x,t) = \begin{cases} u_\nu(t), & x_1 = 1 \\ 0, & \text{otherwise} \end{cases}$$

The greedy algorithm has been applied for the discretized system of dimension $N = 400$ with $\epsilon = 0.05$, and the uniform discretization of $\mathcal{N}$ in $l = 40$.

The offline algorithm stops after 12 iterations, choosing 12 parameter values out of 40 in a zigzag manner.



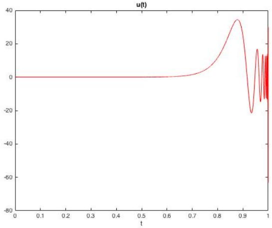
Example 2: Anisotropic 2D Heat Equation-cont.

We aim to steer the system

- from $z_0 = 0$ to $z_1 = \sin(\pi x) * \sin(\pi x_2)$
- in time $T = 1$
- for $\nu = 0.1$

$\Lambda_{\infty,\nu}$ is approximated by a suitable linear combination of $\Lambda_{\infty,i}, i = 1..12$.

Elapsed time is 0.21 s and the error is $|z^1 - z(T)| = 2.0 \times 10^{-4}$.



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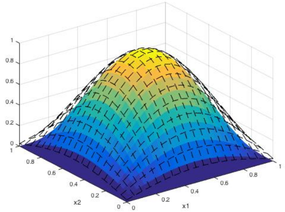

b)

Figure: a) Evolution of the approximate control and b) the states $z(T)$ (dashed) and z^1.

Conclusion

Done:

- Greedy algo for solving parameter dependent OLE
- Provides approximation of infinite time control Gramians (independent of initial and final data, and final time!)
- Enables construction of optimal controls for dissipative systems

Further work:

- Differential Lyapunov equation
- It would provides approximation of finite time control Gramians
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Thanks for your attention!