Optimal control of non-homogeneous parabolic equations via spectral calculus

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Primjer

$$
\begin{aligned}
&\partial_t y = \frac{1}{c\rho}(\lambda \nabla y) - \frac{\sigma}{c\rho} y \text{ na } [0, \infty) \times \Omega, \\
&y(0, \cdot) = u \text{ na } \Omega, \\
&\partial_\nu y = \chi_\Gamma \text{ na } [0, \infty) \times \partial\Omega,
\end{aligned}
$$

models cooling of a steel profile with the geometry $\Omega \subset \mathbb{R}^2$, where c is the specific heat capacity, ρ is the density, λ is the heat conductivity and σ is the surface conductivity. All parameters are functions of the space variables and constant with respect to the time and χ_Γ is the characteristic function of the set $\Gamma \subset \partial\Omega$.

We want to control the system using the function u.

Problem

More precise, we want to solve the following problem:

$$\min_u \{J(u)\colon \|y_u(T) - y^*\| \le \epsilon\},$$

where

$$J(u) = \frac{\alpha}{2}\|u\|^2 + \frac{1}{2}\int_0^T \beta(t)\|y_u(t) - w(t)\|^2 \mathrm{d}t,$$

with $\varepsilon > 0$, $\alpha > 0$, $\beta \in L^\infty((0,T);[0,\infty))$, and $w \in L^2((0,T);\mathcal{H})$.

ε–neighborhood is needed since otherwise in general we would not have a solution.

Abstract version of the problem

$$\begin{cases} y'(t) + Ay(t) + f = 0 \quad \text{za } t \geq 0, \\ y(0) = u. \end{cases}$$

We assume that A is selfadjoint (symmetric) operator.

The solution of the problem is given by the so called Duhamel formula:

$$y_u(t) = S_t u + \int_0^t S_\tau f \,\mathrm{d}\tau,$$

where S_t is the semigroup generated by the operator $-A$ ($S_t = \mathrm{e}^{-At}$).

Solution

(For now) we will assume $f \in \operatorname{Ran} A + \operatorname{Ker} A$ (which is always a dense set).

Using an idea from Lazar, Molinari, Peypouquet: "Optimal control of parabolic equations by spectral decomposition" (2017), one can solve the problem in a following way:

$$\hat{y} = (I + \hat{\gamma} B)^{-1}(\tilde{y}^* + \hat{\gamma} b),$$

where

$$B = \tilde{\alpha} I + \int_0^T \beta(t) S_{2t} \mathrm{d}t, \quad b = \int_0^T \beta(t) S_{T+t} \tilde{w}(t) \mathrm{d}t,$$

and where $\hat{\gamma} > 0$ is the unique solution of

$$G(\gamma) := \|\gamma (I + \gamma B)^{-1}(B\tilde{y}^* - b)\| = \epsilon.$$

The function G is increasing.

Solution - implementation I

Let $g(\gamma) = \gamma(I + \gamma B)^{-1}(B\tilde{y}^* - b)$. Then $G(\gamma) = \|g(\gamma)\|$ and $g(\gamma)$ is the solution of the equation

$$\left(\frac{1}{\gamma} + B\right) x = (B\tilde{y}^* - b).$$

We also have a priori estimates $\hat{\gamma}$ hence $\hat{\gamma}$ can be found by the bisection method.

Once we find $\hat{\gamma}$, the solution of the problem is given by

$$\left(\frac{1}{\hat{\gamma}} + B\right) x = \frac{1}{\hat{\gamma}}\tilde{y}^* + b,$$

Solution - implementation II

How to find (an approximation of) B i b? Using spectral calculus.

$$\int_0^T \beta(t)S_{2t}y\mathrm{d}t = \int_{-\infty}^{\infty}\int_0^T \beta(t)\exp(-2t\lambda)\mathrm{d}t\,\mathrm{d}(E(\lambda)y) = \tilde{\beta}_0(A)y,$$

where $\tilde{\beta}_0(\lambda) = \int_0^T \beta(t)\exp(-2t\lambda)\mathrm{d}t$ and E is the sopectral function of A.

The vector b can be approximated by

$$\tilde{b} = \sum_{i=1}^{N} \tilde{\beta}_i(A)w_i, \text{ where } \tilde{\beta}_i(\lambda) = \int_{t_{i-1}}^{t_i} \beta(t)\exp(-(T+t)\lambda)\mathrm{d}t,$$

if we approximate $\tilde{w}$ by $\sum_{i=1}^{N} w_i\chi_{[t_{i-1},t_i]}$.

Solution - numerical implementation

Operator B and vector b, i.e.. their approximations are calculated using functional calculus via quadrature rules (operator functions are represented as line integrals of functions of the form "scalar function" $\times(\lambda I - A)^{-1}$.

No knowledge of the spectrum of A is necessary!

So far we have implemented a toy 1D example of a non-homogeneous diffusion operator. Looks promising.