Future Research Avenues for Hybrid Systems with Memory

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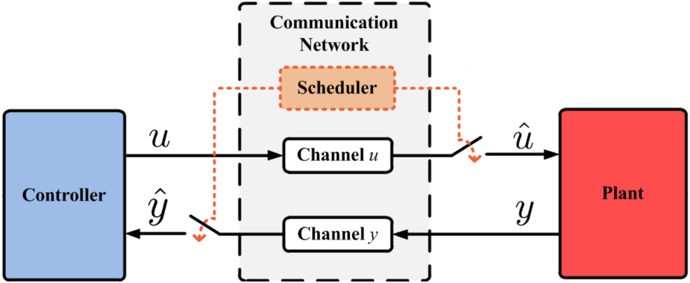
Outline

From Analog to Digital and Networked World

- many processes exhibit both continuous and discrete dynamics – **hybrid dynamics**[1]
- colliding rigid bodies and electrical circuits with active components
- **digital technology** for monitoring and controlling purposes
- communication/measuring devices, sampling units and logic devices
- **delays** in control processes per se and owing to digital technology

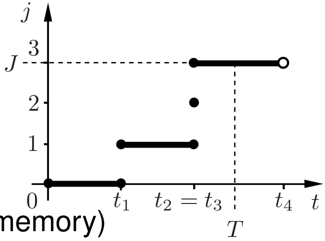
[1] Rafal Goebel, Ricardo G. Sanfelice, and Andrew R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

Real-Life Control System

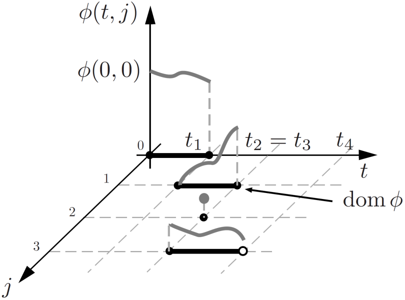


Hybrid Systems with Memory

- hybrid time domain (with memory)



- hybrid arc (with memory)



Hybrid Systems with Memory

- given $d \geq 0$, we use $\mathcal{M}^d$ to denote the **collection of hybrid memory arcs** ϕ satisfying $s \geq -d$ for all $(s, k) \in \mathrm{dom}\ \phi$

- **system state** $x_t : \mathrm{dom}_{\geq 0}(x) \to \mathcal{M}^d$ defined by

$$x_t(s, k(s)) := x(t + s, j + k(s))$$

  for all $(s, k(s)) \in \mathrm{dom}\ x_t$, where $k(s) := \max\{k : (s, k) \in \mathrm{dom}\ x_t\}$ and $\mathrm{dom}\ x_t := \{(s, k) \in \mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0} : (t + s, j + k) \in \mathrm{dom}\ x, s \in [-d, 0]\}$

- d is the maximum value of all delay phenomena in the dynamics

Hybrid Systems with Memory

Definition

A **hybrid system with memory** d is defined by a 4-tuple $\mathcal{H}^d_{\mathcal{M}} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ as follows:

- a set $\mathcal{C} \subset \mathcal{M}^d$, called the **flow set**,
- a function $\mathcal{F} : \mathcal{M}^d \to \mathbb{R}^{n_x}$, called the **flow map**,
- a set $\mathcal{D} \subset \mathcal{M}^d$, called the **jump set**, and
- a function $\mathcal{G} : \mathcal{M}^d \to \mathbb{R}^{n_x}$, called the **jump map.**

[2] Jun Liu and Andrew R. Teel. "Lyapunov-Based Sufficient Conditions for Stability of Hybrid Systems With Memory". In: *IEEE Trans. on Automatic Control* 61.4 (2016), pp. 1057–1062.

Hybrid Systems with Memory

Definition

A hybrid arc is a **solution** to the hybrid system $\mathcal{H}^d_{\mathcal{M}}$ if the **initial data** $x_0 \in \mathcal{C} \cup \mathcal{D}$ and

(i) for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in \mathrm{dom}_{\geq 0}(x)$

$$x_t \in \mathcal{C}, \qquad \dot{x}(t, j) = \mathcal{F}(x_t),$$

(ii) for all $(t, j) \in \mathrm{dom}_{\geq 0}(x)$ such that $(t, j+1) \in \mathrm{dom}_{\geq 0}(x)$,

$$x_t \in \mathcal{D}, \qquad x(t, j+1) = \mathcal{G}(x_t).$$

Set pre-UGAS

Definition

Let $\mathcal{W} \in \mathbb{R}^{n_x}$ be a **closed set.** The set $\mathcal{W}$ is Uniformly Globally pre-Asymptotically Stable (UGpAS) for system $\mathcal{H}^d_{\mathcal{M}}$ if there exists a $\mathcal{KL}$ function β such that any solution x to $\mathcal{H}^d_{\mathcal{M}}$ satisfies

$$\|x(t,j)\|_{\mathcal{W}} \leq \beta\Big(\|x_0\|_{\mathcal{W}}, t+j\Big), \qquad \forall (t,j) \in \mathrm{dom}_{\geq 0}(x),$$

where $\|z\|_{\mathcal{W}} := \inf_{y \in \mathcal{W}} \|y - z\|$ for $z \in \mathbb{R}^{n_x}$ and $\|\phi\|_{\mathcal{W}} := \sup_{\substack{(s,k) \in \mathrm{dom}\,\phi \\ s \in [-d,0]}} \|\phi(s,k)\|_{\mathcal{W}}$ for $\phi \in \mathcal{M}^d$.

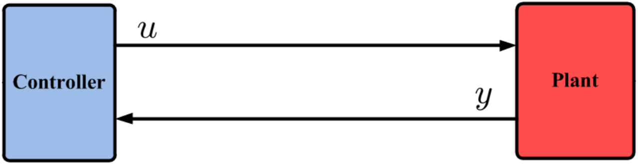
Stabilizing Transmission Intervals and Delays

- nonlinear delayed **plant**

$$\begin{aligned} \dot{x}_p &= f_p(t, x_{p_t}, u_t), \\ y &= g_p(t, x_{p_t}), \end{aligned} \tag{1}$$

- nonlinear delayed **controller**

$$\begin{aligned} \dot{x}_c &= f_c(t, x_{c_t}, y_t), \\ u &= g_c(t, x_{c_t}), \end{aligned} \tag{2}$$



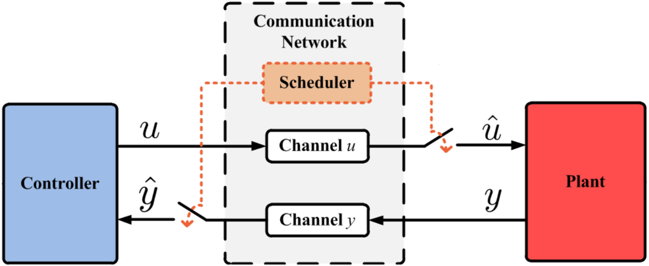
Introduce Communication Network

- ℓ communication channels with discrete delays

$$\big(y(t-d_i(t)), u(t-d_i(t))\big)_i \tag{3}$$

or distributed delays

$$\int_{t-d_i(t)}^{t} \Gamma_i\big(t, s, \big(y(s), u(s)\big)_i\big)\mathrm{d}s, \tag{4}$$



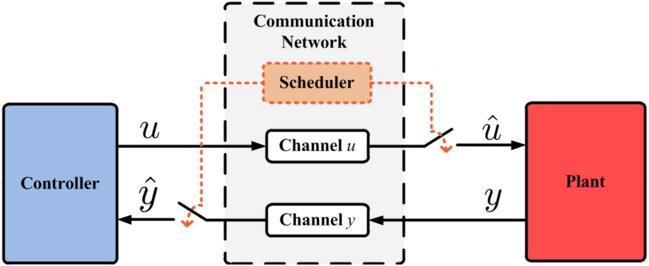
Introduce Communication Network

- nonlinear delayed plant

$$\begin{aligned} \dot{x}_p &= f_p(t, x_{p_t}, \hat{u}_t), \\ y &= g_p(t, x_{p_t}), \end{aligned} \tag{5}$$

- nonlinear delayed controller

$$\begin{aligned} \dot{x}_c &= f_c(t, x_{c_t}, \hat{y}_t), \\ u &= g_c(t, x_{c_t}). \end{aligned} \tag{6}$$

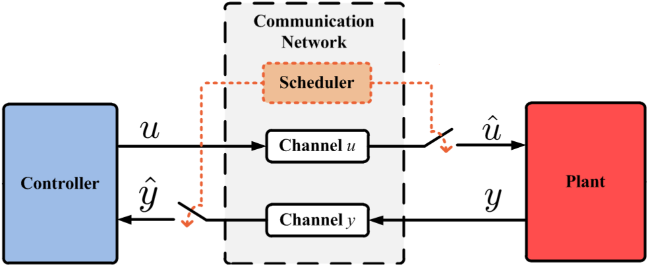


Stabilizing Transmission Intervals and Delays

- define the **error vector**

$$e(t) = \begin{bmatrix} e_y(t) \\ e_u(t) \end{bmatrix} := \begin{bmatrix} \hat{y}(t) - y^*(t) \\ \hat{u}(t) - u^*(t) \end{bmatrix}, \quad (7)$$

where the components of $(y^*(t), u^*(t))$ are given by (3)-(4)



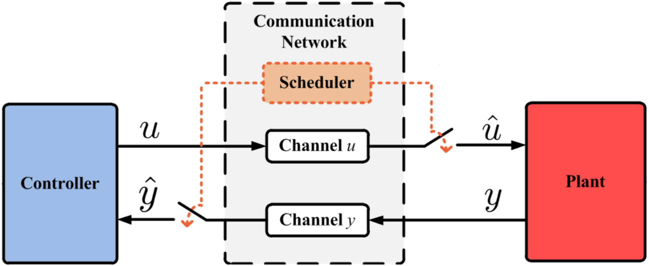
Stabilizing Transmission Intervals and Delays

- $\hat{y}$ and $\hat{u}$ are updated at instants t_κ, $\kappa \in \mathbb{Z}_{\geq 0}$, such that $0 < \epsilon \leq t_{\kappa+1} - t_\kappa \leq \tau_{\mathrm{MATI}}$, yielding

$$\begin{aligned} \hat{y}(t_\kappa^+) &= y^*(t_\kappa) + h_y(\kappa, e(t_\kappa)), \\ \hat{u}(t_\kappa^+) &= u^*(t_\kappa) + h_u(\kappa, e(t_\kappa)), \end{aligned} \tag{8}$$

- consequently

$$e_i(t_\kappa^+) = \mathbf{0}_{n_{e_i}} \tag{9}$$



Hybrid Systems with Memory

- altogether:

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{e} \\ \dot{\tau}_1 \\ \dot{\kappa} \\ \dot{\tau}_2 \end{bmatrix} = \begin{bmatrix} f(x_t, e_t, \tau_{2_t}) \\ g(x_t, e_t, \tau_{2_t}) \\ 1 \\ 0 \\ 1 \end{bmatrix} = \mathcal{F}(\xi_t), \underbrace{\tau_{1_t}(0,0) \in [0, \tau_{\text{MATI}}]}_{\mathcal{C}}, \tag{10}$$

$$\xi^+ = \begin{bmatrix} x^+ \\ e^+ \\ \tau_1^+ \\ \kappa^+ \\ \tau_2^+ \end{bmatrix} = \begin{bmatrix} x_t(0,0) \\ h(\kappa_t(0,0), e_t(0,0)) \\ 0 \\ \kappa_t(0,0) + 1 \\ \tau_{2_t}(0,0) \end{bmatrix} = \mathcal{G}(\xi_t), \underbrace{\tau_{1_t}(0,0) \in [\epsilon, \tau_{\text{MATI}}]}_{\mathcal{D}}, \tag{11}$$

where $\xi := (x, e, \tau_1, \kappa, \tau_2)$

Hybrid Systems with Memory

$$f(x_t, e_t, \tau_{2_t}) \overset{(5),(6)}{:=} \begin{bmatrix} f_p\Big(\tau_2, x_{p_t}, \underbrace{g_{c_t}(\tau_2, x_{c_t}) + e_{u_t}}_{=\hat{u}_t \text{ using (6) and (7)}}\Big) \\ f_c\Big(\tau_2, x_{c_t}, \underbrace{g_{p_t}(\tau_2, x_{p_t}) + e_{y_t}}_{=\hat{y}_t \text{ using (5) and (7)}}\Big) \end{bmatrix} \tag{12}$$

$$h(\kappa_t(0,0), e_t(0,0)) := \begin{bmatrix} h_y\big(\kappa_t(0,0), e_t(0,0)\big) \\ h_u\big(\kappa_t(0,0), e_t(0,0)\big) \end{bmatrix} \tag{13}$$

$$g(x_t, e_t, \tau_{2_t}) \overset{(7)}{:=} \begin{bmatrix} \underbrace{\hat{f}_p\Big(\tau_2, x_{p_t}, x_{c_t}, g_{p_t}(x_{p_t}) + e_{y_t}, g_{c_t}(x_{c_t}) + e_{u_t}\Big)}_{\text{model-based estimator}} - \dot{y}^*(\tau_{2_t}(0,0)) \\ \overbrace{\hat{f}_c\Big(\tau_2, x_{p_t}, x_{c_t}, g_{p_t}(x_{p_t}) + e_{y_t}, g_{c_t}(x_{c_t}) + e_{u_t}\Big)} - \dot{u}^*(\tau_{2_t}(0,0)) \end{bmatrix} \tag{14}$$

Stabilizing Transmission Intervals and Delays

Problem

Find τ_{MATI} that renders the set $\{(x, e, \tau_1, \kappa, \tau_2) : x = \mathbf{0}_{n_x}, e = \mathbf{0}_{n_e}\}$ UGpAS for the closed-loop hybrid dynamics with memory (10)-(11).

Assumptions

Definition

The **protocol** given by $h := (h_y, h_u)$ is **UGES** if there exist a function $W : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, real number $\rho \in [0, 1)$ and functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$ such that the following holds for all $\kappa \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{R}^{n_e}$:

$$\underline{\alpha}(\|e\|) \leq W(\kappa, e) \leq \overline{\alpha}(\|e\|), \tag{15}$$

$$W(\kappa + 1, h(\kappa, e)) \leq \rho W(\kappa, e), \tag{16}$$

Common UGES protocols are Round Robin (RR) and Try Once Discard (TOD).

Assumptions

Assumption

There exist a locally Lipschitz Lyapunov-Krasovskii functional $V : \mathcal{M}^d_{n_x} \to \mathbb{R}_{\geq 0}$, *a continuous functional* $H : \mathcal{M}^d_{n_x} \to \mathbb{R}_{\geq 0}$, *real numbers* $L_i, J_j \geq 0$, *and a continuous positive-definite function* $\varrho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that:*

(i) *for all* $e \in \mathbb{R}^{n_e}$, $\kappa \in \mathbb{Z}_{\geq 0}$, *and* $x_t \in \mathcal{M}^d_{n_x}$ *the inequality*

$$\dot{V}(x_t, \dot{x}_t) \leq -\varrho(\|x_t(0,0)\|) - H^2(x_t) + \gamma^2 W^2(\kappa, e) \tag{17}$$

holds almost everywhere along the solutions of the nominal system $\mathcal{H}_n$, *and*

(ii) *for all nonnegative hybrid memory arcs* $\kappa_t \in \mathcal{M}^d_{n_\kappa}$, *all* $x_t \in \mathcal{M}^d_{n_x}$ *and* $e_t \in \mathcal{M}^d_{n_e}$ *the inequality (18) holds almost everywhere along the solutions of the error system* $\mathcal{H}_e$, *where* $\Delta_i(t)$*'s and* $\delta_j(t)$*'s are differentiable satisfying* $\Delta_i(t) < \overline{\Delta}_i$, $|\dot{\Delta}_i(t)| \leq \breve{\Delta}_i < 1$, $\delta_j(t) < \overline{\delta}_j$ *and* $|\dot{\delta}_j(t)| \leq \breve{\delta}_j < 1$.

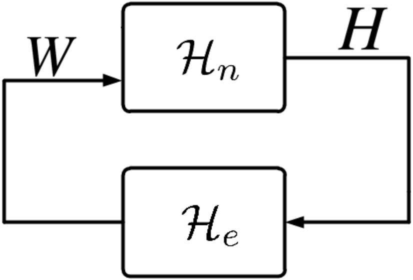
Assumptions

$$\left\langle \frac{\partial W\big(\kappa_t(0,0), e_t(0,0)\big)}{\partial e}, g(x_t, e_t, \tau_{2t}) \right\rangle \leq L_0 W\big(\kappa_t(0,0), e_t(0,0)\big) +$$

$$+ \sum_{i=1}^{n} L_i W\big(\kappa_t(-\Delta_i(t), k(-\Delta_i(t))), e_t(-\Delta_i(t), k(-\Delta_i(t))\big) +$$

$$+ \sum_{j=1}^{m} J_j \int_{t-\delta_j(t)}^{t} W\big(\kappa_t(s, k(s)), e_t(s, k(s))\big) \mathrm{d}s + H(x_t) \tag{18}$$

Assumptions



Main Result

Theorem

Suppose Assumption 1 is satisfied. In addition, suppose that there exist regular enough function $\psi : [0, \tau_{\text{MATI}}] \to \mathbb{R}_{>0}$ and constants $p_i, \mu_i, r_i, \nu_j \geq 0$ such that

(iii) *the inequality*

$$\rho^2 \leq \frac{\psi(\tau_{\text{MATI}})}{\psi(0)}, \text{ and} \tag{19}$$

(iv) *the matrix inequality (20) with expressions (21)-(23)*

*hold, then the set $\mathcal{W} := \{(x, e, \tau_1, \kappa, \tau_2) : x = \mathbf{0}_{n_x}, e = \mathbf{0}_{n_e}\}$ is **UGpAS** for the **closed-loop** system (10)-(11).*

Main Result

$$\begin{bmatrix}
\psi_0 & \psi L_1+L_0L_1\Xi+r_1 & 0 & \cdots & \psi L_n+L_0L_n\Xi+r_n & 0 & \psi J_1+L_0J_1\Xi & \cdots & \psi J_m+L_0J_m\Xi & \psi+L_0\Xi \\
\psi L_1+L_0L_1\Xi+r_1 & \psi_1 & r_1 & \cdots & L_1L_n\Xi & 0 & L_1J_1\Xi & \cdots & L_1J_m\Xi & L_1\Xi \\
0 & r_1 & \overline{\psi}_1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\psi L_n+L_0L_n\Xi+r_n & L_1L_n\Xi & 0 & \cdots & \psi_n & r_n & L_nJ_1 & \cdots & L_nJ_m & L_n\Xi \\
0 & 0 & 0 & \cdots & r_n & \overline{\psi}_n & 0 & \cdots & 0 & 0 \\
\psi J_1+L_0J_1\Xi & L_1J_1\Xi & 0 & \cdots & L_nJ_1 & 0 & \Phi_1 & \cdots & J_1J_m & J_1\Xi \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\psi J_m+L_0J_m\Xi & L_1J_m\Xi & 0 & \cdots & L_nJ_m & 0 & J_1J_m & \cdots & \Phi_m & J_m\Xi \\
\psi+L_0\Xi & L_1\Xi & 0 & \cdots & L_n\Xi & 0 & J_1\Xi & \cdots & J_m\Xi & -1+\Xi
\end{bmatrix} < 0 \tag{20}$$

$$\psi_0 = \gamma^2 + 2\psi L_0 + \dot{\psi} + \sum_{i=1}^{n}(p_i + \mu_i + r_i\overline{\Delta}_i^2 - r_i) + \sum_{j=1}^{m}\nu_j\overline{\delta}_j^2, \tag{21}$$

$$\psi_i = -2r_i + L_i^2\Xi - \mu_i(1-\breve{\Delta}_i) \tag{22}$$

$$\overline{\psi}_i = -p_i - r_i, \qquad \Phi_j = -\nu_j(1-\breve{\delta}_j) + J_j^2\Xi, \qquad \Xi = \sum_{i=1}^{n}\overline{\Delta}_i^2 r_i \tag{23}$$

Example

- plant with delayed input:

$$\dot{x}_p(t) = A_p x_p(t) + B_p u(t - \delta), \quad (24a)$$

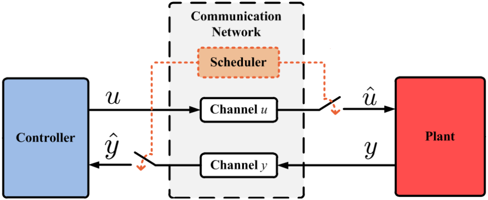
$$y(t) = C_p x_p(t), \quad (24b)$$

- dynamic output feedback observer-predictor controller:

$$\dot{x}_{c1}(t) = A_c x_{c1}(t) + B_c\left[e^{A_p\delta} x_{c2}(t) + \int_{t-\delta}^{t} e^{A_p(t-\theta)} B_p u(\theta) \mathrm{d}\theta\right], \quad (25a)$$

$$\dot{x}_{c2}(t) = A_p x_{c2}(t) + B_p u(t - \delta) + L\left[y(t) - C_p x_{c2}(t)\right], \quad (25b)$$

$$u(t) = C_c x_{c1}(t), \quad (25c)$$

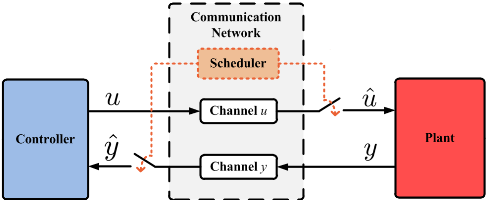


Communication Network

- error vector:

$$e(t) = \begin{bmatrix} e_y(t) \\ e_u(t) \end{bmatrix} = \begin{bmatrix} \hat{y}(t) - \int_{t-\delta}^{t} y(s)\mathrm{d}s \\ \hat{u}(t) - u(t) \end{bmatrix}$$

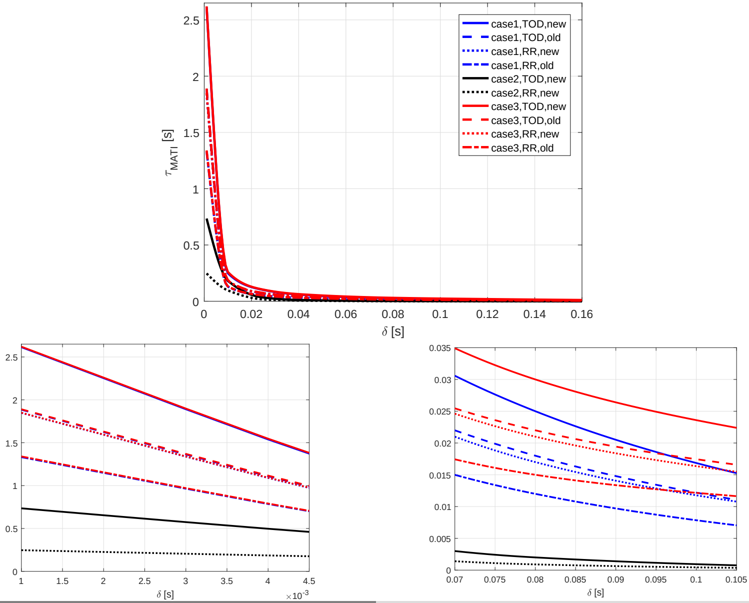
and $\ell = 2$



Three Cases

- **Case 1**: u and y in (24a) and (25b), respectively, are replaced with $\hat{u}$ and $\hat{y}$, respectively,
- **Case 2**: Case 1 together with $\hat{u}$ instead of u in (25a)-(25b), and
- **Case 3**: Case 2 augmented with the model-based predictor $\dot{\hat{u}} = C_c B_c \int_{t-\delta}^{t} e^{A_p(t-\theta)} B_p \hat{u}(\theta) \mathrm{d}\theta$.

Numerical Results



Future Work

- more generic Lypunov-Krasovskii functionals
- optimal parameters in the presented stability conditions
- event- and self-triggering

Thanks

This work has been supported by Croatian Science Foundation under the project IP-2016-06-2468 “ConDyS”.

Questions? Comments? Suggestions?

Questions? Comments?
Suggestions?

Thank You for Your attention!!