Greedy Algorithm for Parameter Dependent Operator Lyapunov Equations

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Background and Motivation

Consider a family of parameter-dependent linear control problems

$$\begin{cases} \frac{d}{dt}x_\nu(t) = A_\nu x_\nu(t) + B_\nu u_\nu(t), & 0 \leq t \leq T \\ x_\nu(0) = x_{0,\nu} \end{cases} \quad (1)$$

where

- $A_\nu : D(A_\nu) \subset X \to X$.
- $B_\nu : U \to X_{-1}$.
- Parameter $\nu \in \mathcal{N} \subset \mathbb{R}^n$ and $\mathcal{N}$ is compact.

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If x_1 is a desired final state, for $\epsilon > 0$, we wish to design u such that $||x_1 - x(T)|| < \epsilon$.

Background and Motivation

One way to construct u in order to achieve this result is to set $u = B^*\phi$, where ϕ is the solution to the adjoint problem

$$\begin{cases} -\frac{d}{dt}\phi(t) = A^*\phi(t), & 0 \leq t \leq T \\ \phi(T) = \phi_0 \end{cases} \tag{2}$$

where ϕ_0 is the solution to the following associated optimal control problem

$$\min_{z \in R_T \cap \partial B_\epsilon(x^1)} J(z) = \left\{ \frac{1}{2} ||B^*\phi||^2 : \phi(T) = \Lambda_T^{-1}(z - e^{TA}x_0) \right\}. \tag{3}$$

Background and Motivation

We compute Λ_T as

$$\Lambda_T = \int_0^T e^{tA} BB^* e^{tA^*}\, dt.$$

It is known that for sufficiently large T, the controllability Gramian can be approximated by

$$\Lambda_\infty = \int_0^\infty e^{tA} BB^* e^{tA^*}\, dt.$$

This approximation is the solution to the (algebraic) operator Lyapunov equation (OLE)

$$AP + PA^* = -BB^*.$$

Background and Motivation

We consider the related family of parameter-dependent operator Lyapunov equations (OLE)

$$A_\nu P_\nu + P_\nu A_\nu^* = -B_\nu B_\nu$$

and we wish to establish an efficient greedy algorithm for constructing an approximate controllability Gramian to use to build the necessary optimal control.

Framework: Required Spaces

For Hilbert space X and unbounded linear operator A let X_1 be the $D(A)$ equipped with the norm

$$||x||_1 = ||(\beta I - A)x||, \quad x \in D(A), \beta \in \rho(A)$$

and X_{-1} be X completed with the norm

$$||x||_{-1} = ||(\beta I - A)^{-1}x||, \quad x \in X, \beta \in \rho(A).$$

Similar spaces X_1^d and X_{-1}^d can be made by replacing A with its adjoint.

Framework: Assumptions on $\mathcal{A} = \{A_\nu\}$, $\mathcal{A}^* = \{A_\nu^*\}$

For each operator A_ν:

- A_ν is closed.
- A_ν is stable.
- A_ν is sectorial, i.e. there exists constants $\omega < 0,\ \theta \in (\frac{\pi}{2}, \pi),\ M > 0$ such that

$$\begin{cases} \rho(A_\nu) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ \|(\lambda I - A_\nu)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|},\ \forall \lambda \in S_{\theta,\omega}. \end{cases}$$

- $D(A_\nu)$ is dense in X.
- $D(A_{\nu_1}) = D(A_{\nu_2})$ and $D(A^*_{\nu_1}) = D(A^*_{\nu_2})$ for $\nu_1,\ \nu_2 \in \mathcal{N}$.

Framework: Consequences

For semigroup $\mathbb{T} = \{e^{tA_\nu}\}_{t \geq 0}$

- $\mathbb{T}$ is strongly continuous.
- $\mathbb{T}$ is strongly stable.
- $\mathbb{T}$ is an analytic semigroup.

Framework: Assumptions on $\mathcal{B} = \{B_\nu\}$

Each operator B_ν is an infinite-time admissible control operator for semigroup $\mathbb{T}$, i.e. for every $u \in L^2([0,\infty), U)$, the mapping $\tau \to \Phi_\tau u$ is bounded in X. The map Φ is defined as

$$\Phi_\tau u := \int_0^\tau \mathbb{T}_t B_\nu u(t) dt.$$

Framework: Sufficient Conditions

Theorem

Let $\mathbb{T}$ be a strongly continuous semigroup on the Hilbert space X, with generator A. Then the following statements are equivalent:

- *B is an infinite-time admissible control operator for $\mathbb{T}$.*
- *There exists an operator $P \in \mathcal{L}(X)$ such that for any $x \in X_1^d$,*

$$Px = \lim_{\tau \to \infty} \int_0^\tau \mathbb{T}_t BB^* \mathbb{T}_t^* x dt.$$

- *There exist operators $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$, which satisfies (OLE) with BB^*, $A\Pi + \Pi A^* \in \mathcal{L}(X_1^d, X_{-1})$.*

Moreover, if (1) holds and $\mathbb{T}^*$ is strongly stable, then P is the unique self-adjoint solution of (OLE)

Greedy Algorithm: Outline

Algorithm 1 (Weak) Greedy Algorithm

Initialize: Fix a constant $\gamma \in (0, 1]$ and $\epsilon > 0$;

1: In the first step, choose $P_1 \in \mathcal{P}$ such that

$$||P_1||_{\mathbb{P}} \geq \gamma \max_{P \in \mathcal{P}} ||P||_{\mathbb{P}}.$$

2: At the general step, having found $P_1, \cdots, P_k$, denote

$$\mathcal{P}_k = \text{span}\{P_1, \cdots, P_k\} \text{ and } \sigma_k(\mathcal{P}) := \max_{P \in \mathcal{P}} \text{dist}(P, \mathcal{P}_k);$$

3: **repeat**

4: choose P_{k+1} such that

$$\text{dist}(P_{k+1}, \mathcal{P}_k) \geq \gamma\, \sigma_k(\mathcal{P});$$

5: **until** $\sigma_k(\mathcal{P}) < \epsilon$;

Greedy Algorithm: Offline

Fix the approximation error $\epsilon > 0$.

- STEP 1: (discretization) Choose a finite subset $\tilde{\mathcal{N}}$ such that for all $\nu \in \mathcal{N}$,
$$\text{dist}(\nu, \tilde{\mathcal{N}}) < \delta$$
where $\delta > 0$.
- STEP 2: Choosing ν_1
Choose $\nu_1 \in \tilde{\mathcal{N}}$ in some manner.
Compute P_1 as the solution of (OLE) corresponding to $\nu = \nu_1$

Greedy Algorithm: Offline

- STEP 3: Choosing ν_{j+1}
  Suppose the last chosen parameter is ν_j. Calculate $R_\nu(P_j) = L_{A_\nu}(P_j) + B_\nu B_\nu^*$ for each $\nu \in \tilde{\mathcal{N}}$.
  Check the inequality

$$\max_{\nu \in \tilde{\mathcal{N}}} \inf_{P \in \mathcal{P}_j} ||R_\nu(P)||_{op} < \frac{\epsilon}{2}$$

  where $\mathcal{P}_j = \text{span}\{P_1, \cdots, P_j\}$.
  If the inequality is satisfied, stop the algorithm. Else, determine ν_{j+1} as

$$\nu_{j+1} \in \arg\max_{\nu \in \tilde{\mathcal{N}}} \inf_{P \in \mathcal{P}_j} ||R_\nu(P)||_{op}.$$

  Find the solution to OLE $P_{\nu_{j+1}}$ and repeat Step 3.

Greedy Algorithm: Online

Parameter value $\nu \in \mathcal{K}$ is given.

- STEP 1: Project $-B_\nu B_\nu^*$ to $L_{A_\nu}(\mathcal{P}_k) :=$ span $\{L_{A_\nu}(P_j)\}_{j=1}^n$ and denote the projection by $\Pi_l(-B_\nu B_\nu^*)$.
- STEP 2: Solve for α_j

$$\Pi_l(-B_\nu B_\nu^*) = \Sigma \alpha_j L_{A_\nu}(P_j)$$

- STEP 3: Define approximating Gramian operator as

$$\Lambda_{\nu,a} := \Sigma \alpha_j P_j$$

Residual Analysis: The space $\mathcal{L}(X_1^d, X_{-1})$

We can easily show that $R_\nu(P_j) \in \mathcal{L}(X_{1,\nu}^d, X_{-1,\nu})$ for each ν.
Note that

$$||R_\nu(P)||_{op} = \sup\{||R_\nu(P)z||_{-1,\nu} : ||z||_{1d,\nu} = 1,\ z \in X_{1,\nu}^d\}.$$

As mentioned previously, the domains of all operators in $\mathcal{A}^*$ are the same. Thus X_{1,ν_1}^d and X_{1,ν_2}^d differ only with respect to their norms.
If the respective norms are equivalent, we drop parameter dependence on $\mathcal{L}(X_{1,\nu}^d, X_{-1,\nu})$

Example 1: Perturbed 2D Heat Equation

$$
\begin{aligned}
&\tfrac{\partial}{\partial t} z - \Delta_\nu z = 0 && \text{in} && (0,1)^2 \times (0,T), \\
&z(x,t) = v_0(x,t), && \text{for} && x \in \partial([0,1]^2) \\
&z(x,0) = z_0(x)
\end{aligned}
\tag{4}
$$

$$
\Delta_\nu = \frac{\partial^2}{\partial x_1^2} + (1+\nu)\frac{\partial^2}{\partial x_2^2}, \qquad \nu \in \mathcal{N} = [0,1]
$$

Example 1: Perturbed 2D Heat Equation

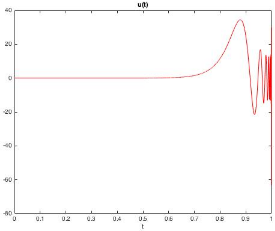
What we refer to as "right-boundary" is the set
$\partial_R = \{(x_1, x_2) \in \partial([0,1]^2) : x_1 = 1\}$.

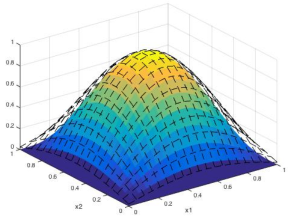
Take

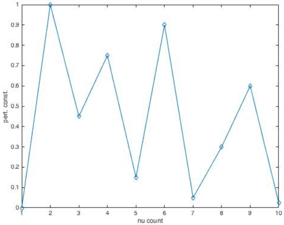
$$v_0(x,t) = \begin{cases} u_\nu(t), & x \in \partial_R \\ 0, & \text{otherwise} \end{cases}$$

We aim to control the system to final state
$z_1 = \sin(\pi x_1) * \sin(\pi x_2)$ in time $T = 1$ with $\nu = 0.1$. The greedy algorithm has been applied for the discretized system of dimension $N = 400$ with $\epsilon = 0.2$, and the uniform discretization of $\mathcal{N}$ in $k = 41$.

Example 1: Plots





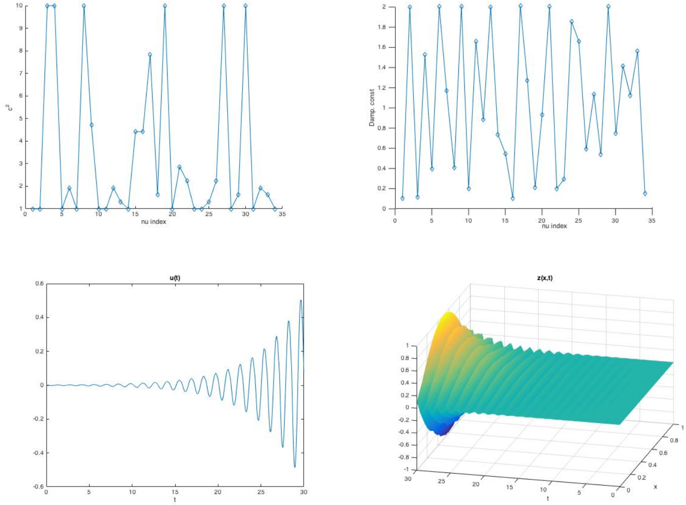


Example 2: 1D Damped Wave Equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} z - c^2 \frac{\partial^2}{\partial x^2} z = -\beta \frac{\partial}{\partial t} z & \text{in} \quad (0,1) \times (0,T), \\ z(0,t) = v_0(t), & z(1,t) = v_1(t) \\ z(x,0) = z_0(x), & z_t(x,0) = z_{0*}(x). \end{cases} \tag{5}$$

Take $v_0(t) = 0$ and allow $v_1(t) = u_\nu(t)$ to be our right-boundary control with final state $z_1 = \sin(\pi x)$ in time $T = 30$ with $\nu = (2, 0.4)$. Here $N = 40$ with $\epsilon = 0.01$, and the discretization of $\mathcal{N}$ in $k = 1200$.

Example 2: Plots



Discussion