Greedy Algorithm for Parameter Dependent Operator Lyapunov Equations

Martin Lazar, Jerome Weston

University of Dubrovnik

International Conference on Elliptic and Parabolic Problems
Hotel Serapo, Gaeta-Italy

May 22, 2019

ConDyS

hrzz
Hrvatska zaklada za znanost

Table of Contents

Background and Motivation

Consider a family of parameter-dependent linear control problems

$$\begin{cases} \frac{d}{dt}x_\nu(t) = A_\nu x_\nu(t) + B_\nu u_\nu(t), & 0 \le t \le T \\ x_\nu(0) = x_{0,\nu} \end{cases} \quad (1)$$

where parameter $\nu \in \mathcal{N} \subset \mathbb{R}^n$ and $\mathcal{N}$ is compact.

Background and Motivation

Consider a family of parameter-dependent linear control problems

$$\begin{cases} \frac{d}{dt}x_\nu(t) = A_\nu x_\nu(t) + B_\nu u_\nu(t), \quad 0 \leq t \leq T \\ x_\nu(0) = x_{0,\nu} \end{cases} \tag{1}$$

where parameter $\nu \in \mathcal{N} \subset \mathbb{R}^n$ and $\mathcal{N}$ is compact.

If x_1 is a desired final state, for $\epsilon > 0$, we wish to design u such that $||x_1 - x(T)|| < \epsilon$.

Background and Motivation

One way to construct u in order to achieve this result is to set $u = B^*\phi$, where ϕ is the solution to the adjoint problem

$$\begin{cases} -\frac{d}{dt}\phi(t) = A^*\phi(t), & 0 \le t \le T \\ \phi(T) = \phi_0 \end{cases} \quad (2)$$

where ϕ_0 is the solution to the following associated optimal control problem

$$\min_{z \in R_T \cap \partial B_\epsilon(x^1)} J(z) = \left\{ \frac{1}{2} ||B^*\phi||^2 : \phi(T) = \Lambda_T^{-1}(z - e^{TA}x_0) \right\}. \quad (3)$$

Background and Motivation

We compute Λ_T as

$$\Lambda_T = \int_0^T e^{tA} B B^* e^{tA^*} \, dt.$$

It is known that for sufficiently large T and A stable, the controllability Gramian can be approximated by

$$\Lambda_\infty = \int_0^\infty e^{tA} B B^* e^{tA^*} \, dt.$$

This approximation is the solution to the (algebraic) operator Lyapunov equation (OLE)

$$AP + PA^* = -BB^*.$$

Background and Motivation

We consider the related family of parameter-dependent operator Lyapunov equations (OLE)

$$A_\nu P_\nu + P_\nu A_\nu^* = -B_\nu B_\nu$$

and we wish to establish an efficient greedy algorithm for constructing an approximate controllability Gramian to use to build the necessary optimal control with minimum L^2-norm.

Framework: Required Spaces

For Hilbert space X and unbounded linear operator A let X_1 be the $D(A)$ equipped with the norm

$$||x||_1 = ||(\beta I - A)x||, \quad x \in D(A), \beta \in \rho(A)$$

and X_{-1} be X completed with the norm

$$||x||_{-1} = ||(\beta I - A)^{-1}x||, \quad x \in X, \beta \in \rho(A).$$

Similar spaces X_1^d and X_{-1}^d can be made by replacing A with its adjoint.

Framework: Assumptions on $\mathcal{A} = \{A_\nu\},\ \mathcal{A}^* = \{A_\nu^*\}$

For each operator $A_\nu : X_{1,\nu} \to X$:

1) A_ν is closed.
2) A_ν is stable.
3) A_ν is sectorial [1], i.e. there exists constants $\omega \in \mathbb{R},\ \theta \in (\frac{\pi}{2}, \pi),\ M > 0$ such that

$$\begin{cases} \rho(A_\nu) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ \|(\lambda I - A_\nu)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|},\ \forall \lambda \in S_{\theta,\omega}. \end{cases}$$

4) $D(A_\nu)$ is dense in X.
5) $D(A_{\nu_1}) = D(A_{\nu_2})$ and $D(A^*_{\nu_1}) = D(A^*_{\nu_2})$ for $\nu_1,\ \nu_2 \in \mathcal{N}$.

---

[1] Lunardi, A, *Analytic Semigroups and Optimal Regularity in Parabolic Problems,*

Framework: Consequences

Lemma

If A_ν has properties 1-4, then $\mathbb{T} = \{e^{tA_\nu}\}_{t\geq 0}$ is a strongly continuous, strongly stable, and analytic semigroup.

Framework: Assumptions on $\mathcal{B} = \{B_\nu\}$

Each operator $B_\nu : U \to X_{-1,\nu}$ is an infinite-time admissible control operator for semigroup $\mathbb{T}$, i.e. for every $u \in L^2([0,\infty), U)$, the mapping $\tau \to \Phi_\tau u$ is bounded in X. The map Φ is defined as

$$\Phi_\tau u := \int_0^\tau \mathbb{T}_t B_\nu u(t) dt.$$

Framework: Sufficient Conditions

Theorem (Hansen-Weiss (1997))

Let $\mathbb{T}$ be a strongly continuous semigroup on the Hilbert space X, with generator A. If B is an infinite-time admissible control operator for $\mathbb{T}$, then there exists an operator $P \in \mathcal{L}(X)$ such that

1) *For any $x \in X_1^d$, $Px = \lim_{\tau \to \infty} \int_0^\tau \mathbb{T}_t BB^* \mathbb{T}_t^* x \, dt$.*

2) *$P \geq 0$ satisfies (OLE) with BB^*, $AP + PA^* \in \mathcal{L}(X_1^d, X_{-1})$.*

Moreover, if $\mathbb{T}^*$ is strongly stable, then P is the unique self-adjoint solution of (OLE)

Greedy Algorithm: Idea

Approximate (compact) solution manifold $\mathcal{P}$ in a Banach space $\mathbb{P}$ by a sequence of finite dimensional subspaces $\mathcal{P}_k$ of dimension k.

Offline procedure generates approximation subspace within given precision error; Online routine calculates approximations for any element in $\mathcal{P}$.

Greedy Algorithm: Offline Outline

Algorithm 1 (Weak) Greedy Algorithm

Initialize: Fix a constant $\gamma \in (0, 1]$ and $\epsilon > 0$;

1: In the first step, choose $P_1 \in \mathcal{P}$ such that

$$||P_1||_{\mathbb{P}} \geq \gamma \max_{P \in \mathcal{P}} ||P||_{\mathbb{P}}.$$

2: At the general step, having found $P_1, \cdots, P_k$, denote

$$\mathcal{P}_k = \text{span}\{P_1, \cdots, P_k\} \text{ and } \sigma_k(\mathcal{P}) := \max_{P \in \mathcal{P}} \text{dist}(P, \mathcal{P}_k);$$

3: **repeat**

4: choose P_{k+1} such that

$$\text{dist}(P_{k+1}, \mathcal{P}_k) \geq \gamma \, \sigma_k(\mathcal{P});$$

5: **until** $\sigma_k(\mathcal{P}) < \epsilon$;

Greedy Algorithm: Distance Approximation

Theorem ((DeVore, Petrova, Wojtasczyk - 2013))

For the weak greedy algorithm with constant γ *in a Hilbert space* $\mathbb{P}$, *we have the following: If the compact set* $\mathcal{P}$ *is such that, for some* $\alpha > 0$ *and* $C_0 > 0$

$$d_k(\mathcal{P}) \leq C_0 n^{-\alpha} \ , \ n \in \mathbb{N},$$

then

$$\sigma_k(\mathcal{P}) \leq C_1 n^{-\alpha} \ , \ n \in \mathbb{N},$$

where $C_1 := \gamma^{-2} 2^{5\alpha+1} C_0$.

Greedy Algorithm: Distance Approximation

Theorem ((DeVore, Petrova, Wojtasczyk - 2013))

For the weak greedy algorithm with constant γ in a Hilbert space $\mathbb{P}$, we have the following: If the compact set $\mathcal{P}$ is such that, for some $\alpha > 0$ and $C_0 > 0$

$$d_k(\mathcal{P}) \leq C_0 n^{-\alpha}\ ,\ n \in \mathbb{N},$$

then

$$\sigma_k(\mathcal{P}) \leq C_1 n^{-\alpha}\ ,\ n \in \mathbb{N},$$

where $C_1 := \gamma^{-2} 2^{5\alpha+1} C_0$.

For our purposes, we choose $d_k(\mathcal{P})$ as the *Kolmogorov n-width of $\mathcal{P}$* defined as

$$d_k(\mathcal{P}) := \inf_{dim Y = k} \sup_{P \in \mathcal{P}} \inf_{\Gamma \in Y} ||P - \Gamma||_{\mathbb{P}}$$

Implementation: Residual Analysis

In computation we take

$$d_k(\mathcal{P}) = \max_{\nu \in \tilde{\mathcal{N}}} \inf_{P \in \mathcal{P}_k} ||R_\nu(P)||_{\mathcal{L}(X^d_{1,\nu}, X_{-1,\nu})}$$

where

- $\tilde{\mathcal{N}}$ is a discrete subset of $\mathcal{N}$ such that for all $\nu \in \mathcal{N}$, $\text{dist}(\nu, \tilde{\mathcal{N}}) < \delta$ for some $\delta > 0$.
- $R_\nu(P) := A_\nu P - PA_\nu + B_\nu B_\nu^*$

Implementation: Residual Analysis

We can easily show that $R_\nu(P_j) \in \mathcal{L}(X^d_{1,\nu}, X_{-1,\nu})$ for each ν.
As mentioned previously, the domains of all operators in $\mathcal{A}^*$ are the same. Thus X^d_{1,ν_1} and X^d_{1,ν_2} differ only with respect to their norms.
If the respective norms are equivalent, we drop parameter dependence on $\mathcal{L}(X^d_{1,\nu}, X_{-1,\nu})$

Implementation: Residual Analysis

We can easily show that $R_\nu(P_j) \in \mathcal{L}(X^d_{1,\nu}, X_{-1,\nu})$ for each ν.
As mentioned previously, the domains of all operators in $\mathcal{A}^*$ are the same. Thus X^d_{1,ν_1} and X^d_{1,ν_2} differ only with respect to their norms.
If the respective norms are equivalent, we drop parameter dependence on $\mathcal{L}(X^d_{1,\nu}, X_{-1,\nu})$

Theorem

$$||R||_{\mathcal{L}(X^d_1, X_{-1})} \sim ||P - P_\nu||_{\mathcal{L}(X^d_1, X)}$$

Implementation: Online Routine

For specific $\nu \in \mathcal{N}$

- Project $-B_\nu B_\nu^*$ to $L_\nu(\mathcal{P}_k) :=$ span $\{L_\nu(P_j)\}_{j=1}^k$.
  Here $L_\nu(P_j) = A_\nu P_j + P_j A_\nu^*$.
- Define approximating Gramian operator as $\Lambda_{\nu,a} := \Sigma \alpha_j P_j$

The corresponding approximate control is computed as before.

Example: 1D Damped Wave Equation

$$\left\{ \begin{array}{rcl} \frac{\partial^2}{\partial t^2} z - c^2 \frac{\partial^2}{\partial x^2} z = -\beta \frac{\partial}{\partial t} z & \text{in} & (0,1) \times (0,T), \\ z(0,t) = 0, & & z(1,t) = u_\nu(t) \\ z(x,0) = 0, & & \frac{\partial}{\partial t} z(x,0) = 0. \end{array} \right. \tag{4}$$

Take final state $z_1 = \sin(\pi x)$.
Here $N = 40$ with $\epsilon = 0.01$, and the discretization of $\mathcal{N} = [1, 10] \times [0.1, 2]$ of $I = 1200$.

Example: Plots

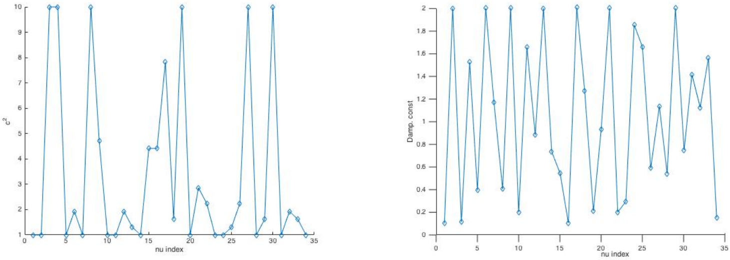

Figure: Distribution of selected parameter values for a) velocity squared and b) dampening constant.

Example: Plots

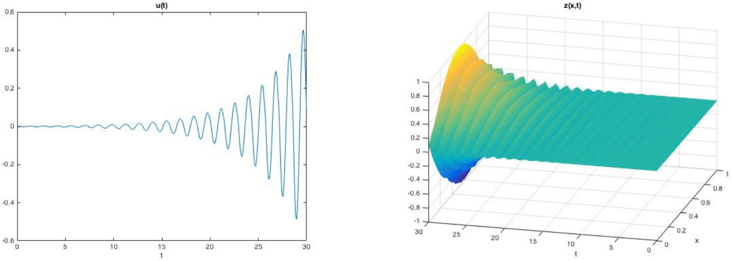

Figure: Evolution of a) the approximate control and b) the solution of semi-discretized example problem governed by the approximate control for $\nu = (2,\ 0.4)$ in time T = 30.

References

- DeVore, R., G. Petrova, and P.Wojtasczyk, *Greedy Algorithms for Reduced Bases in Banach Spaces,* Constr. Approx., 37 (2013) 455-466.
- Hansen, S. and G. Weiss,*New results on the operator Carleson measure criterion,* IMA J. Math. Control Inform., 14 (1997), pp. 3-32. Distributed parameter systems: analysis, synthesis and applications, Part 1.
- Lunardi, A, *Analytic Semigroups and Optimal Regularity in Parabolic Problems,* Birkhauser-Verlag, Switzerland, 1995
- Tucsnak, M. and G. Weiss, *Observation and Control for Operator Semigroups,* Birkhauser-Verlag AG, Berlin, 2009.