Remarks on Lyapunov operator in infinite dimensional setting

Martin Lazar
University of Dubrovnik

International Conference on Elliptic and Parabolic Problems
Gaeta, May 2019

ConDyS

Hrvatska zaklada za znanost

Lyapunov Equation

Lyapunov Equation

$$AP + PA^* = -Q,$$

where A, Q **are given** operators and P **is an unknown**.
Applications to

- stability of systems
- control theory

LHS is the Lyapunov operator

$$L_A(P) = AP + PA^*.$$

We are interested in properties of $L_A(P)$:

- boundedness,
- coercivity, ...?

Finite dimensional case

If

- X - finite dimensional Hilbert space
- A - a continuous linear operator on X

the results follow directly.

$$(\forall P \in \mathcal{L}(X)) \quad |L_A(P)| = |AP + PA^*| \le |A||P| \qquad (\textit{boundedness})$$

In addition, if A is stable

$$P = -\int_0^\infty e^{tA} L_A(P) e^{tA^*} dt$$

implying

$$|P| \le |L_A(P)| \int_0^\infty e^{-\omega t}, \quad \omega > 0. \qquad (\textit{coercivity})$$

Infinite dimensional setting

- X - Hilbert space
- A - unbounded operator on X with dense domain.

For simplicity of presentation

- A self adjoint with compact resolvent (e.g. $-\Delta$).

$$(\psi_k, \lambda_k) \qquad -\text{sequence of eigenpairs.}$$

What can we say about L_A in this case?
Is it

- well defined on $\mathcal{L}(X)$,
- bounded,
- coercive?

Eigenproperties

Spectrum

$$\sigma(L_A) = \{(\lambda_i + \lambda_j), i, j \in \mathbf{N}\}$$

Corresponding eigenoperators E_{ij} are defined by

$$E_{ij}\psi_k = \psi_i \delta_{jk}.$$

Obviously, L_A is not bounded on $\mathcal{L}(X)$.
Indeed,

$$L_A(E_{ij}) = \underbrace{(\lambda_i + \lambda_j)}_{\longrightarrow_i \infty} E_{ij}$$

Even more, for an $P \in \mathcal{L}(X)$ arbitrary, $L_A(P)$ is not in $\mathcal{L}(X)$ at all.

$$\begin{aligned} L_A(P) = AP + PA &- \text{not defined on } X \\ &- \text{but on } D(A)! \end{aligned} \tag{1}$$

Functional setting

Define spaces

- X_1 - as $D(A)$ equipped with the norm:

$$\|x\|_1 = \|(\beta I - A)x\|, \quad \beta \in \rho(A).$$

- X_{-1} - completion of X with respect to the norm

$$\|x\|_{-1} = \|(\beta I - A)^{-1}x\|,$$

  - independent of $\beta \in \rho(A)$.

It holds

$$X_1 \hookrightarrow X \hookrightarrow X_{-1}$$

and X_{-1} is dual to X_1. [1]
Moreover:
– $A \in \mathcal{L}(X_1, X)$
– A has a unique extension $A \in \mathcal{L}(X, X_{-1})$
It is easy to see that for every $P \in \mathcal{L}(X)$

$$L_A(P) = AP + PA \in \mathcal{L}(X_1; X_{-1})$$

[1] M. Tucsnak and G. Weiss, Observation and Control for Operator Semigroups, Birkhauser-Verlag AG, Berlin, 2009.

Boundedness

Theorem 1.

For every $P \in Sym(X)$

$$\|L_A(P)\|_{\mathcal{L}(X_1;X_{-1})} \leq \frac{2}{|\lambda_1|}\|P\|_{\mathcal{L}(X)},$$

where λ_1 is the smallest (or largest) eigenvalue of A.

Dem. $\{E_{ij} + E_{ji}, i, j \in \mathbf{N}\}$ - basis of $Sym(X)$.

$$\begin{aligned}\|E_{ij} + E_{ji}\|_{\mathcal{L}(X_1,X_{-1})} &= \sup_k \|(\beta I - A)^{-1}(E_{ij} + E_{ji})\tilde{\psi}_k\| \\ &= \frac{1 + \delta_{ij}}{|\beta - \lambda_i||\beta - \lambda_j|}.\end{aligned}$$

where $\tilde{\psi}_k = \psi_k/(\beta - \lambda_k)$ denotes the orthonormal basis in X_1 .

$$\begin{aligned}\|L_A(E_{ij} + E_{ji})\|_{\mathcal{L}(X_1,X_{-1})} &= \|(\lambda_i + \lambda_j)(E_{ij} + E_{ji})\|_{\mathcal{L}(X_1,X_{-1})} \\ &= \left|\frac{(\lambda_i + \lambda_j)(1 + \delta_{ij})}{(\beta - \lambda_i)(\beta - \lambda_j)}\right| \leq \frac{2}{\lambda_1} \underbrace{(1 + \delta_{ij})}_{=\|E_{ij}+E_{ji}\|_{\mathcal{L}(X)}} \qquad (2)\end{aligned}$$

Q.ED.

Coercivity fails

The above functional setting does not imply coercivity.

Example:

$\|\lambda_i E_{ii}\|_{\mathcal{L}(X)} \to \infty$

$\|L_A(\lambda_i E_{ii})\|_{\mathcal{L}(X_1;X_{-1})} = \frac{2\lambda_i^2}{|\beta-\lambda_i|^2} \leq 2$

We need to relax the norm on P!

Coercivity

Theorem 2.

Suppose that either A or $-A$ is a positive operator. Then for every $P \in Sym(X)$ it holds

$$\frac{1}{2}\|P\|_{\mathcal{L}(X_1,X)} \leq \|L_A(P)\|_{\mathcal{L}(X_1,X_{-1})} \leq 2\|P\|_{\mathcal{L}(X_1,X)} \,.$$

Dem.

$$\|E_{ii}\|_{\mathcal{L}(X_1,X)} \approx \frac{1}{\lambda_i} \quad \text{(before it was 1)}$$

and

$$\|L_A(E_{ii})\|_{\mathcal{L}(X_1,X_{-1})} \approx \frac{1}{\lambda_i}$$

Q.ED.

Control theory

Infinite control Gramian

$$\Lambda_\infty = \int_0^\infty e^{tA} BB^* e^{tA} dt$$

is the solution to Lyapunov equation

$$AP + PA = -BB^*,$$

where $B : U \to X_{-1}$ is a control operator.
We have

$$-BB^* = L_A(\Lambda_\infty)$$

The last theorem implies

$$\|BB^*\|_{\mathcal{L}(X_1, X_{-1})} \sim 2\|\Lambda_\infty\|_{\mathcal{L}(X_1, X)} .$$

Example:

A - the heat operator on Ω
$B = \chi_\omega, \quad \omega \subseteq \Omega$
The eigenvalues of Λ_∞ decay exponentially as $\exp(-c\sqrt{\lambda_k})$ (bounds both from below[2] and above[1]).

[1] L. Grubišić and D. Kressner, On the eigenvalue decay of solutions to operator Lyapunov equations, Systems & Control Letters, 73 (2014) 4247.

[2] E. Zuazua, Eigenvalue bounds for the Grammian operator of the heat equation, preprint 2016.

- Can the spectrum of Λ_∞ corresponding to different operators A be related to the properties of Lyapunov operator?
- Can the presented results be generalised by reducing the assumptions on A? To which extent?

Thanks for your attention!