Efficient Approximation of Bounds for Parameter Dependent QEP

Zoran Tomljanović

University J. J. Strossmayer of Osijek
Department of Mathematics
Trg Ljudevita Gaja 6
31000 Osijek, Croatia
http://www.mathos.unios.hr

ztomljan@mathos.hr

Joint work with:

Matea Puvača, Ninoslav Truhar

[2019, Osijek, Croatia] 10.5.2019.

Overview

1. Introduction
   - Problem formulation
   - Optimization criterion
2. Eigenvalue approximation
   - First order approximation
     - Numerical example
   - Application
     - Numerical example

Introduction

Consider a damped linear vibrating system

$$M\ddot{q} + D\dot{q} + Kq = 0,$$
$$q(0) = q_0, \quad and \quad \dot{q}(0) = \dot{q}_0,$$

where $M, D, K \in \mathbb{R}^{n \times n}$, symmetric (mass, damping, stiffness),

M and $K > 0$ positive definite.

$D = C_{int} + C_{ext}$, $C_{int} > 0$ internal damping, $C_{ext} \geq 0$ external damping.

$C_{int} = \alpha_c C_{crit}$, where $C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}$.

Introduction

Consider a damped linear vibrating system

$$M\ddot{q} + D\dot{q} + Kq = 0,$$
$$q(0) = q_0, \quad and \quad \dot{q}(0) = \dot{q}_0,$$

where $M, D, K \in \mathbb{R}^{n\times n}$, symmetric (mass, damping, stiffness), M and $K > 0$ positive definite.

$D = C_{int} + C_{ext}$, $C_{int} > 0$ internal damping, $C_{ext} \geq 0$ external damping.

$C_{int} = \alpha_c C_{crit}$, where $C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}$.

Introduction

Consider a damped linear vibrating system

$$M\ddot{q} + D\dot{q} + Kq = 0,$$
$$q(0) = q_0, \quad and \quad \dot{q}(0) = \dot{q}_0,$$

where $M, D, K \in \mathbb{R}^{n\times n}$, symmetric (mass, damping, stiffness),

M and $K > 0$ positive definite.

$D = C_{int} + C_{ext}, C_{int} > 0$ internal damping, $C_{ext} \geq 0$ external damping.

$C_{int} = \alpha_c C_{crit}$, where $C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}$.

Introduction

Consider a damped linear vibrating system

$$M\ddot{q} + D\dot{q} + Kq = 0,$$
$$q(0) = q_0, \quad and \quad \dot{q}(0) = \dot{q}_0,$$

where $M, D, K \in \mathbb{R}^{n\times n}$, symmetric (mass, damping, stiffness), M and $K > 0$ positive definite.

$D = C_{int} + C_{ext}$, $C_{int} > 0$ internal damping, $C_{ext} \geq 0$ external damping.

$C_{int} = \alpha_c C_{crit}$, where $C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}$.

Example

Consider *n-mass oscillator or oscillator ladder*

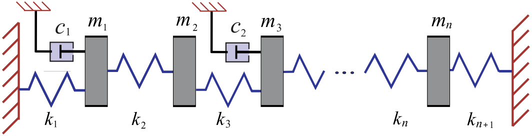


Figure: The n-mass oscillator with two dampers

$m_i > 0$ - masses, c_1, c_2 - viscosities, $k_i > 0$ - stiffnesses

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n), D \equiv C_{int} + C_{ext} = C_{int} + c_1 e_1 e_1^T + c_2 e_3 e_3^T,$$

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix}.$$

Example

Consider *n-mass oscillator or oscillator ladder*

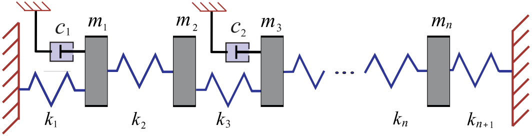


Figure: The n-mass oscillator with two dampers

$m_i > 0$ - masses, c_1, c_2 - viscosities, $k_i > 0$ - stiffnesses

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n), D \equiv C_{int} + C_{ext} = C_{int} + c_1 e_1 e_1^T + c_2 e_3 e_3^T,$$

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix}.$$

Example

Consider *n-mass oscillator or oscillator ladder*

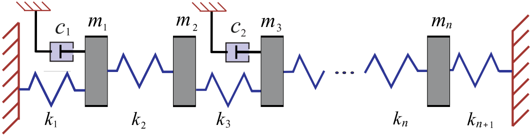


Figure: The n-mass oscillator with two dampers

$m_i > 0$ - masses, c_1, c_2 - viscosities, $k_i > 0$ - stiffnesses

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n), D \equiv C_{int} + C_{ext} = C_{int} + c_1 e_1 e_1^T + c_2 e_3 e_3^T,$$

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix}.$$

Linearization

- Let Φ simultaneously diagonalize pair M and K

$$\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I.$$

Note that internal damping is s.t. $\Phi^T C_u \Phi = \alpha\Omega$.
With $q = \Phi q_\Phi$ and $w_1 = \Omega q_\Phi$, $w_2 = \dot{q}_\Phi$ we have

$$\frac{d}{dt}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D \Phi \end{bmatrix}}_{A} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

We obtain first order differential equation:

$$\dot{w} = Aw, \quad \text{with solution} \quad w = e^{At} w_0, \quad \text{where } w_0 \text{ is initial data.}$$

Linearization

- Let Φ simultaneously diagonalize pair M and K

$$\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I.$$

Note that internal damping is s.t. $\Phi^T C_u \Phi = \alpha\Omega$.
With $q = \Phi q_\Phi$ and $w_1 = \Omega q_\Phi$, $w_2 = \dot{q}_\Phi$ we have

$$\frac{d}{dt}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D \Phi \end{bmatrix}}_{A} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

We obtain first order differential equation:

$\dot{w} = Aw$, with solution $w = e^{At} w_0$, where w_0 is initial data.

Linearization

- Let Φ simultaneously diagonalize pair M and K

$$\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I.$$

Note that internal damping is s.t. $\Phi^T C_u \Phi = \alpha\Omega$.
With $q = \Phi q_\Phi$ and $w_1 = \Omega q_\Phi$, $w_2 = \dot{q}_\Phi$ we have

$$\frac{d}{dt}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D \Phi \end{bmatrix}}_{A} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

We obtain first order differential equation:

$$\dot{w} = Aw, \quad \text{with solution} \quad w = e^{At} w_0, \quad \text{where } w_0 \text{ is initial data.}$$

Very important question arises in considering such systems:

For the given mass (M) and stiffness (K) determine the best (optimal) damping which will insure optimal evanescence.

(M, K), A not stable

(M, K, D), A stable

Very important question arises in considering such systems:

For the given mass (M) and stiffness (K) determine the best (optimal) damping which will insure optimal evanescence.

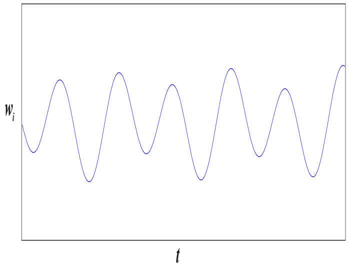
(M, K), A not stable

(M, K, D), A stable

Very important question arises in considering such systems:

For the given mass (M) and stiffness (K) determine the best (optimal) damping which will insure optimal evanescence.

(M, K), A not stable

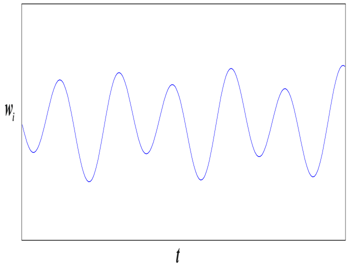


(M, K, D), A stable

Very important question arises in considering such systems:

For the given mass (M) and stiffness (K) determine the best (optimal) damping which will insure optimal evanescence.

(M, K), A not stable

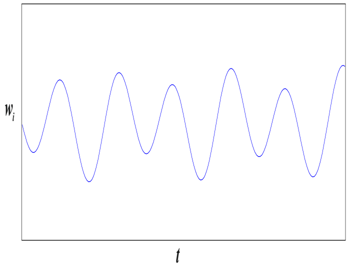


(M, K, D), A stable

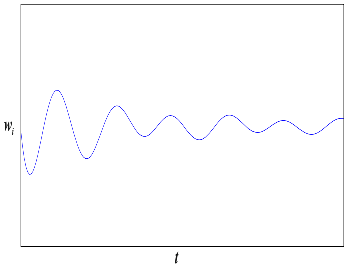
Very important question arises in considering such systems:

For the given mass (M) and stiffness (K) determine the best (optimal) damping which will insure optimal evanescence.

(M, K), A not stable



(M, K, D), A stable



Optimization criterion

There are different optimization criteria, but we use:

- Spectral abscissa criterion:

$$\max_k \mathrm{Re}\left(\lambda_k(\mathbf{v})\right) \quad \to \quad min,$$

where $\lambda_k(\mathbf{v})$ are the eigenvalues of

$$\left(\lambda^2(\mathbf{v})M(\mathbf{v}) + \lambda(\mathbf{v})D(\mathbf{v}) + K(\mathbf{v})\right)x(\mathbf{v}) = 0.$$

- Frequency isolation criterion.

Optimization criterion

There are different optimization criteria, but we use:

- Spectral abscissa criterion:

$$\max_k \operatorname{Re}\left(\lambda_k(\mathbf{v})\right) \quad \rightarrow \quad min,$$

where $\lambda_k(\mathbf{v})$ are the eigenvalues of

$$\left(\lambda^2(\mathbf{v})M(\mathbf{v}) + \lambda(\mathbf{v})D(\mathbf{v}) + K(\mathbf{v})\right) x(\mathbf{v}) = 0.$$

- Frequency isolation criterion.

Our goal

We have $(\lambda_k(\mathbf{v^0}), x_k(\mathbf{v^0}))$ for $k = 1, \ldots, 2n$.

We would like to efficiently calculate approximations of eigensystem $(\lambda_k(\mathbf{v}), x_k(\mathbf{v}))$ for parameter $\mathbf{v}$ such that

$$\|\mathbf{v} - \mathbf{v^0}\| \ll \delta_v.$$

$$\frac{\partial \lambda_k}{\partial v_i}(\mathbf{v}) = -\frac{y_k^*(\mathbf{v})\left(\lambda_k^2(\mathbf{v})\frac{\partial M}{\partial v_i}(\mathbf{v}) + \lambda_k(\mathbf{v})\frac{\partial D}{\partial v_i}(\mathbf{v}) + \frac{\partial K}{\partial v_i}(\mathbf{v})\right)x_k(\mathbf{v})}{y_k^*(\mathbf{v})(2\lambda_k(\mathbf{v})M(\mathbf{v}) + D(\mathbf{v}))x_k(\mathbf{v})}.$$

Our goal

We have $(\lambda_k(\mathbf{v^0}), x_k(\mathbf{v^0}))$ for $k = 1, \ldots, 2n$.

We would like to efficiently calculate approximations of eigensystem $(\lambda_k(\mathbf{v}), x_k(\mathbf{v}))$ for parameter $\mathbf{v}$ such that

$$\|\mathbf{v} - \mathbf{v^0}\| \ll \delta_v.$$

$$\frac{\partial \lambda_k}{\partial v_i}(\mathbf{v}) = -\frac{y_k^*(\mathbf{v})\left(\lambda_k^2(\mathbf{v})\frac{\partial M}{\partial v_i}(\mathbf{v}) + \lambda_k(\mathbf{v})\frac{\partial D}{\partial v_i}(\mathbf{v}) + \frac{\partial K}{\partial v_i}(\mathbf{v})\right)x_k(\mathbf{v})}{y_k^*(\mathbf{v})(2\lambda_k(\mathbf{v})M(\mathbf{v}) + D(\mathbf{v}))x_k(\mathbf{v})}.$$

Theorem

Let $\lambda_k(\mathbf{v})$ denotes function $\lambda_k : \mathbb{R}^s_+ \to \mathbb{R}$ that corresponds to the k-th eigenvalue. Moreover let $\mathbf{H}(\lambda_k(\mathbf{v}))$ be the Hessian matrix of function $\lambda_k(\mathbf{v})$. For $\mathbf{v} \in \Omega_{\mathbf{v}^0} = \{\mathbf{v} \in \mathbb{R}^s_+ : ||\mathbf{v} - \mathbf{v}^0|| \leq \delta_{\mathbf{v}}\}$, where $\delta_{\mathbf{v}}$ is given tolerance, approximation $\tilde{\tilde{\lambda}}_k(\mathbf{v})$ can be calculated using

$$\tilde{\tilde{\lambda}}_k(\mathbf{v}) = \lambda_k(\mathbf{v}^0) + \sum_{i=1}^{s} \frac{\partial \lambda_k}{\partial v_i}(\mathbf{v}^0)(v_i - v_i^0)$$

and for error bound it holds

$$|\lambda_k(\mathbf{v}) - \tilde{\tilde{\lambda}}_k(\mathbf{v})| \lessapprox \frac{1}{2} M_k \|(\mathbf{v} - \mathbf{v}^0)\|^2,$$

for $M_k = \max_{t \in [0,1]} (\|\mathbf{H}(\lambda_k(\mathbf{v}^0 + t(\mathbf{v} - \mathbf{v}^0))\|)$.

Theorem

Let $\lambda_k(\mathbf{v})$ *denotes function* $\lambda_k : \mathbb{R}^s_+ \to \mathbb{R}$ *that corresponds to the* k*-th eigenvalue. Moreover let* $\mathbf{H}(\lambda_k(\mathbf{v}))$ *be the Hessian matrix of function* $\lambda_k(\mathbf{v})$*. For* $\mathbf{v} \in \Omega_{\mathbf{v}^0} = \{\mathbf{v} \in \mathbb{R}^s_+ : ||\mathbf{v} - \mathbf{v}^0|| \leq \delta_{\mathbf{v}}\}$*, where* $\delta_{\mathbf{v}}$ *is given tolerance, approximation* $\tilde{\tilde{\lambda}}_k(\mathbf{v})$ *can be calculated using*

$$\tilde{\tilde{\lambda}}_k(\mathbf{v}) = \lambda_k(\mathbf{v}^0) + \sum_{i=1}^{s} \frac{\partial \lambda_k}{\partial v_i}(\mathbf{v}^0)(v_i - v_i^0)$$

and for error bound it holds

$$|\lambda_k(\mathbf{v}) - \tilde{\tilde{\lambda}}_k(\mathbf{v})| \lessapprox \frac{1}{2} M_k \|(\mathbf{v} - \mathbf{v}^0)\|^2,$$

for $M_k = \max_{t \in [0,1]} (\|\mathbf{H}(\lambda_k(\mathbf{v}^0 + t(\mathbf{v} - \mathbf{v}^0))\|)$*.*

$$H(\lambda_k(\mathbf{u}))_{i,j} = \frac{\partial^2 \lambda_k}{\partial v_i \partial v_j}(\mathbf{u}) =$$

$$-\frac{y_k^*(\mathbf{u})\left(\frac{\partial\lambda_k}{\partial v_i}(\mathbf{u})\frac{\partial\lambda_k}{\partial v_j}(\mathbf{u})2M(\mathbf{u}) + Z_{ij}(\mathbf{u})\right)x_k(\mathbf{u})}{y_k^*(\mathbf{u})(2\lambda_k(\mathbf{u})M(\mathbf{u}) + D(\mathbf{u}))x_k(\mathbf{u})}$$

$$-\frac{y_k^*(\mathbf{u})\left(\lambda_k^2(\mathbf{u})\frac{\partial^2 M}{\partial v_i \partial v_j}(\mathbf{u}) + \lambda_k(\mathbf{u})\frac{\partial^2 D}{\partial v_i \partial v_j}(\mathbf{u}) + \frac{\partial^2 K}{\partial v_i \partial v_j}(\mathbf{u})\right)x_k(\mathbf{u})}{y_k^*(\mathbf{u})(2\lambda_k(\mathbf{u})M(\mathbf{u}) + D(\mathbf{u}))x_k(\mathbf{u})}$$

$$-\frac{y_k^*(\mathbf{u})(N_i(\mathbf{u})PI_k(\mathbf{u})N_j(\mathbf{u}) + N_j(\mathbf{u})PI_k(\mathbf{u})N_i(\mathbf{u}))x_k(\mathbf{u})}{y_k^*(\mathbf{u})(2\lambda_k(\mathbf{u})M(\mathbf{u}) + D(\mathbf{u}))x_k(\mathbf{u})},$$

where

$$\begin{aligned} Z_{ij}(\mathbf{u}) &= \frac{\partial\lambda_k}{\partial v_i}(\mathbf{u})\left(2\lambda_k(\mathbf{u})\frac{\partial M}{\partial v_j}(\mathbf{u}) + \frac{\partial D}{\partial v_j}(\mathbf{u})\right) + \\ &\quad \frac{\partial\lambda_k}{\partial v_j}(\mathbf{u})\left(2\lambda_k(\mathbf{u})\frac{\partial M}{\partial v_i}(\mathbf{u}) + \frac{\partial D}{\partial v_i}(\mathbf{u})\right), \\ N_l(\mathbf{u}) &= 2\lambda_k(\mathbf{u})\frac{\partial\lambda_k}{\partial v_l}(\mathbf{u})M(\mathbf{u}) + \frac{\partial\lambda_k}{\partial v_l}(\mathbf{u})D(\mathbf{u}) + \\ &\quad \lambda_k^2(\mathbf{u})\frac{\partial M}{\partial v_l}(\mathbf{u}) + \lambda_k(\mathbf{u})\frac{\partial D}{\partial v_l}(\mathbf{u}) + \frac{\partial K}{\partial v_l}(\mathbf{u}), \\ PI_k(\mathbf{u}) &= -\left(\lambda_k^2(\mathbf{u})M(\mathbf{u}) + \lambda_k(\mathbf{u})D(\mathbf{u}) + K(\mathbf{u})\right)^{\dagger}. \end{aligned}$$

$$H(\lambda_k(\mathbf{u}))_{i,j} = \frac{\partial^2 \lambda_k}{\partial v_i \partial v_j}(\mathbf{u}) =$$

$$-\frac{y_k^*(\mathbf{u})\left(\frac{\partial \lambda_k}{\partial v_i}(\mathbf{u})\frac{\partial \lambda_k}{\partial v_j}(\mathbf{u})2M(\mathbf{u}) + Z_{ij}(\mathbf{u})\right)x_k(\mathbf{u})}{y_k^*(\mathbf{u})(2\lambda_k(\mathbf{u})M(\mathbf{u}) + D(\mathbf{u}))x_k(\mathbf{u})}$$

$$-\frac{y_k^*(\mathbf{u})\left(\lambda_k^2(\mathbf{u})\frac{\partial^2 M}{\partial v_i \partial v_j}(\mathbf{u}) + \lambda_k(\mathbf{u})\frac{\partial^2 D}{\partial v_i \partial v_j}(\mathbf{u}) + \frac{\partial^2 K}{\partial v_i \partial v_j}(\mathbf{u})\right)x_k(\mathbf{u})}{y_k^*(\mathbf{u})(2\lambda_k(\mathbf{u})M(\mathbf{u}) + D(\mathbf{u}))x_k(\mathbf{u})}$$

$$-\frac{y_k^*(\mathbf{u})(N_i(\mathbf{u})PI_k(\mathbf{u})N_j(\mathbf{u}) + N_j(\mathbf{u})PI_k(\mathbf{u})N_i(\mathbf{u}))x_k(\mathbf{u})}{y_k^*(\mathbf{u})(2\lambda_k(\mathbf{u})M(\mathbf{u}) + D(\mathbf{u}))x_k(\mathbf{u})},$$

where

$$\begin{aligned}
Z_{ij}(\mathbf{u}) &= \frac{\partial \lambda_k}{\partial v_i}(\mathbf{u})\left(2\lambda_k(\mathbf{u})\frac{\partial M}{\partial v_j}(\mathbf{u}) + \frac{\partial D}{\partial v_j}(\mathbf{u})\right) + \\
&\quad \frac{\partial \lambda_k}{\partial v_j}(\mathbf{u})\left(2\lambda_k(\mathbf{u})\frac{\partial M}{\partial v_i}(\mathbf{u}) + \frac{\partial D}{\partial v_i}(\mathbf{u})\right), \\
N_l(\mathbf{u}) &= 2\lambda_k(\mathbf{u})\frac{\partial \lambda_k}{\partial v_l}(\mathbf{u})M(\mathbf{u}) + \frac{\partial \lambda_k}{\partial v_l}(\mathbf{u})D(\mathbf{u}) + \\
&\quad \lambda_k^2(\mathbf{u})\frac{\partial M}{\partial v_l}(\mathbf{u}) + \lambda_k(\mathbf{u})\frac{\partial D}{\partial v_l}(\mathbf{u}) + \frac{\partial K}{\partial v_l}(\mathbf{u}), \\
PI_k(\mathbf{u}) &= -\left(\lambda_k^2(\mathbf{u})M(\mathbf{u}) + \lambda_k(\mathbf{u})D(\mathbf{u}) + K(\mathbf{u})\right)^\dagger .
\end{aligned}$$

Numerical example

Example 1 Let dimension be $n = 100$, and

$$
\begin{aligned}
M(\mathbf{v}^0) &= \operatorname{diag}(2, 4, \ldots, 200), \\
K(\mathbf{v}^0) &= 0.1 \cdot \begin{bmatrix} 4 & -1 & -1 & & & \\ -1 & 4 & -1 & -1 & & \\ -1 & -1 & 4 & -1 & -1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & -1 & -1 & 4 & -1 & -1 \\ & & -1 & -1 & 4 & -1 \\ & & & -1 & -1 & 4 \end{bmatrix} + v_3^0 \cdot \tilde{K},
\end{aligned}
$$

where

$$
\tilde{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, K0 = \tilde{K}(30:55, 30:55) = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}.
$$

Numerical example

Example 1

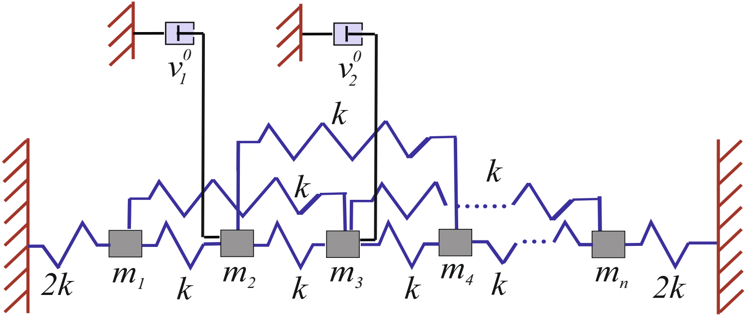


Figure: n-mass oscillator

$D(\mathbf{v}^0) = \alpha C_{crit} + C_{ext}(\mathbf{v}^0)$, where $\alpha = 0.004$,

$$C_{ext}(\mathbf{v}^0) = v_1^0 e_{35} e_{35}^T + v_2^0 e_{50} e_{50}^T.$$

Parameters $\mathbf{v}$ and $\mathbf{v^0}$ are as following

$\mathbf{v} = (\frac{c}{4} + \delta, c + \delta, \delta)$ and $\mathbf{v^0} = (\frac{c}{4}, c, 0)$.

$D(\mathbf{v}^0) = \alpha C_{crit} + C_{ext}(\mathbf{v}^0)$, where $\alpha = 0.004$,

$$C_{ext}(\mathbf{v}^0) = v_1^0 e_{35} e_{35}^T + v_2^0 e_{50} e_{50}^T.$$

Parameters $\mathbf{v}$ and $\mathbf{v^0}$ are as following

$$\mathbf{v} = (\tfrac{c}{4} + \delta, c + \delta, \delta) \text{ and } \mathbf{v^0} = (\tfrac{c}{4}, c, 0).$$

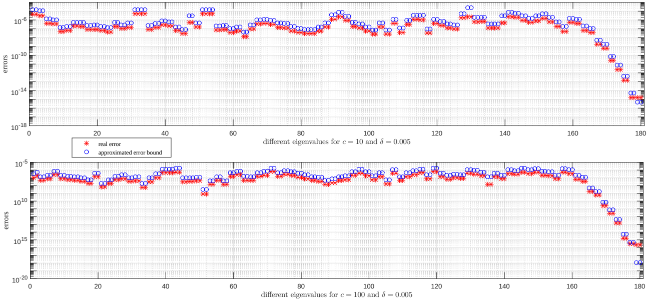


Figure: Comparison of real error $|\lambda(\mathbf{v}) - \widetilde{\widetilde{\lambda}}(\mathbf{v})|$ and approximated error bound of eigenvalues, for system where $c = 10$ and $\delta = 0.005$ and for system where $c = 100$ and $\delta = 0.005$

$\sin\Theta$ type theorem for the eigensubspaces for quadratic hyperbolic eigenvalue problem

[N.Truhar, S. Miodragović,'15.]

Additional assumption:

$$(x^*D(\mathbf{v^0})x)^2 - 4(x^*M(\mathbf{v^0})x)(x^*K(\mathbf{v^0})x) > 0, \quad \text{for all } x \text{ of PQEP.}$$

$$(M(\mathbf{v^0}), D(\mathbf{v^0}), K(\mathbf{v^0})) \xrightarrow{linearization} (A(\mathbf{v^0}), J(\mathbf{v^0}))$$

(X, Λ) eigenpair of $(A(\mathbf{v^0}), J(\mathbf{v^0}))$

$$(M(\mathbf{v}), D(\mathbf{v}), K(\mathbf{v})) \xrightarrow{linearization} (A(\mathbf{v}), J(\mathbf{v}))$$

$(\widetilde{X}, \widetilde{\Lambda})$ eigenpair of $(A(\mathbf{v}), J(\mathbf{v}))$

$\sin\Theta$ type theorem for the eigensubspaces for quadratic hyperbolic eigenvalue problem

[N.Truhar, S. Miodragović,'15.]

Additional assumption:

$$(x^*D(\mathbf{v^0})x)^2 - 4(x^*M(\mathbf{v^0})x)(x^*K(\mathbf{v^0})x) > 0, \quad \text{for all } x \text{ of PQEP.}$$

$$(M(\mathbf{v^0}), D(\mathbf{v^0}), K(\mathbf{v^0})) \xrightarrow{linearization} (A(\mathbf{v^0}), J(\mathbf{v^0}))$$

$$(X, \Lambda) \text{ eigenpair of } (A(\mathbf{v^0}), J(\mathbf{v^0}))$$

$$(M(\mathbf{v}), D(\mathbf{v}), K(\mathbf{v})) \xrightarrow{linearization} (A(\mathbf{v}), J(\mathbf{v}))$$

$$(\widetilde{X}, \widetilde{\Lambda}) \text{ eigenpair of } (A(\mathbf{v}), J(\mathbf{v}))$$

$\sin\Theta$ type theorem for the eigensubspaces for quadratic hyperbolic eigenvalue problem

[N.Truhar, S. Miodragović,'15.]

Additional assumption:

$$(x^*D(\mathbf{v^0})x)^2 - 4(x^*M(\mathbf{v^0})x)(x^*K(\mathbf{v^0})x) > 0, \quad \text{for all } x \text{ of PQEP.}$$

$$(M(\mathbf{v^0}), D(\mathbf{v^0}), K(\mathbf{v^0})) \xrightarrow{linearization} (A(\mathbf{v^0}), J(\mathbf{v^0}))$$

$$(X, \Lambda) \text{ eigenpair of } (A(\mathbf{v^0}), J(\mathbf{v^0}))$$

$$(M(\mathbf{v}), D(\mathbf{v}), K(\mathbf{v})) \xrightarrow{linearization} (A(\mathbf{v}), J(\mathbf{v}))$$

$$(\widetilde{X}, \widetilde{\Lambda}) \text{ eigenpair of } (A(\mathbf{v}), J(\mathbf{v}))$$

$$|\sin\vartheta(X(:,j),\widetilde{X}(:,j))| \leq \kappa(X)\kappa(\widetilde{X})\left(\frac{\delta a_F+\delta c_F}{\mathrm{RelGap}_{2,j}}+\frac{\delta b_F}{\mathrm{RelGap}_{1,j}}\right)$$

$$:= RHS_{\mathrm{RelGap}},$$

where

$$\mathrm{RelGap}_{1,j}=\min_{i\neq j}\frac{|\lambda_j-\widetilde{\lambda}_i|}{|\widetilde{\lambda}_i|},\quad \mathrm{RelGap}_{2,j}=\min_{i\neq j}\frac{|\lambda_j-\widetilde{\lambda}_i|}{|\lambda_j|},$$

and

$$\delta a_F=\sqrt{\|L_K^* M(\mathbf{v^0})^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2+\|L_M^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2},$$

$$\delta b_F=\sqrt{\|L_M^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2+\|L_K^{-1}(K(\mathbf{v})-K(\mathbf{v^0}))L_K^{-*}\|_F^2},$$

$$\delta c_F=\sqrt{\|L_M^{-1}(D(\mathbf{v})-D(\mathbf{v^0}))L_K^{-*}+L_M^{-1}D(\mathbf{v^0})M(\mathbf{v^0})^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2}.$$

$$|\sin\vartheta(X(:,j),\widetilde{X}(:,j))| \leq \kappa(X)\kappa(\widetilde{X})\left(\frac{\delta a_F+\delta c_F}{\mathrm{RelGap}_{2,j}}+\frac{\delta b_F}{\mathrm{RelGap}_{1,j}}\right)$$
$$:= RHS_{\mathrm{RelGap}},$$

where

$$\mathrm{RelGap}_{1,j}=\min_{i\neq j}\frac{|\lambda_j-\widetilde{\lambda}_i|}{|\widetilde{\lambda}_i|},\quad \mathrm{RelGap}_{2,j}=\min_{i\neq j}\frac{|\lambda_j-\widetilde{\lambda}_i|}{|\lambda_j|},$$

and

$$\delta a_F=\sqrt{\|L_K^* M(\mathbf{v^0})^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2+\|L_M^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2},$$
$$\delta b_F=\sqrt{\|L_M^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2+\|L_K^{-1}(K(\mathbf{v})-K(\mathbf{v^0}))L_K^{-*}\|_F^2},$$
$$\delta c_F=\sqrt{\|L_M^{-1}(D(\mathbf{v})-D(\mathbf{v^0}))L_K^{-*}+L_M^{-1}D(\mathbf{v^0})M(\mathbf{v^0})^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2}.$$

$$|\sin\vartheta(X(:,j),\widetilde{X}(:,j))| \leq \kappa(X)\kappa(\widetilde{X})\left(\frac{\delta a_F+\delta c_F}{\mathrm{RelGap}_{2,j}}+\frac{\delta b_F}{\mathrm{RelGap}_{1,j}}\right)$$
$$:= RHS_{\mathrm{RelGap}},$$

where

$$\mathrm{RelGap}_{1,j}=\min_{i\neq j}\frac{|\lambda_j-\widetilde{\lambda}_i|}{|\widetilde{\lambda}_i|},\quad \mathrm{RelGap}_{2,j}=\min_{i\neq j}\frac{|\lambda_j-\widetilde{\lambda}_i|}{|\lambda_j|},$$

and

$$\delta a_F=\sqrt{\|L_K^{*}M(\mathbf{v^0})^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2+\|L_M^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2},$$
$$\delta b_F=\sqrt{\|L_M^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2+\|L_K^{-1}(K(\mathbf{v})-K(\mathbf{v^0}))L_K^{-*}\|_F^2},$$
$$\delta c_F=\sqrt{\|L_M^{-1}(D(\mathbf{v})-D(\mathbf{v^0}))L_K^{-*}+L_M^{-1}D(\mathbf{v^0})M(\mathbf{v^0})^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2}.$$

$$|\sin\vartheta(X(:,j),\widetilde{X}(:,j))| \leq \kappa(X)\kappa(\widetilde{X})\left(\frac{\delta a_F+\delta c_F}{\mathrm{RelGap}_{2,j}}+\frac{\delta b_F}{\mathrm{RelGap}_{1,j}}\right) := RHS_{\mathrm{RelGap}},$$

where

$$\mathrm{RelGap}_{1,j}=\min_{i\neq j}\frac{|\lambda_j-\widetilde{\lambda}_i|}{|\widetilde{\lambda}_i|}, \quad \mathrm{RelGap}_{2,j}=\min_{i\neq j}\frac{|\lambda_j-\widetilde{\lambda}_i|}{|\lambda_j|},$$

and

$$\delta a_F=\sqrt{\|L_K^{*}M(\mathbf{v^0})^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2+\|L_M^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2},$$

$$\delta b_F=\sqrt{\|L_M^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2+\|L_K^{-1}(K(\mathbf{v})-K(\mathbf{v^0}))L_K^{-*}\|_F^2},$$

$$\delta c_F=\sqrt{\|L_M^{-1}(D(\mathbf{v})-D(\mathbf{v^0}))L_K^{-*}+L_M^{-1}D(\mathbf{v^0})M(\mathbf{v^0})^{-1}(M(\mathbf{v})-M(\mathbf{v^0}))L_M^{-*}\|_F^2}.$$

$$\mathrm{RelGap}_{1,j} \geq \min_{i\neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{|\widetilde{\widetilde{\lambda}}_i(\mathbf{v})| + M_{i_1}} =: \mathrm{BoundRelGap}_{1,j},$$

$$\mathrm{RelGap}_{2,j} \geq \min_{i\neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{\lambda_j(\mathbf{v^0})} =: \mathrm{BoundRelGap}_{2,j},$$

where $M_{i_1} = \frac{1}{2} M_i \|(\mathbf{v} - \mathbf{v}^0)\|^2$.

$$RHS_{\mathrm{BoundRelGap}} = \kappa(X)\kappa(\widetilde{X}) \left(\frac{\delta a_F + \delta c_F}{\mathrm{BoundRelGap}_{2,j}} + \frac{\delta b_F}{\mathrm{BoundRelGap}_{1,j}} \right)$$

It holds,

$$RHS_{\mathrm{RelGap}} \leq RHS_{\mathrm{BoundRelGap}}.$$

$$\mathrm{RelGap}_{1,j} \geq \min_{i \neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{|\widetilde{\widetilde{\lambda}}_i(\mathbf{v})| + M_{i_1}} =: \mathrm{BoundRelGap}_{1,j},$$

$$\mathrm{RelGap}_{2,j} \geq \min_{i \neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{\lambda_j(\mathbf{v^0})} =: \mathrm{BoundRelGap}_{2,j},$$

where $M_{i_1} = \frac{1}{2} M_i \|(\mathbf{v} - \mathbf{v}^0)\|^2$.

$$RHS_{\mathrm{BoundRelGap}} = \kappa(X)\kappa(\widetilde{X}) \left(\frac{\delta a_F + \delta c_F}{\mathrm{BoundRelGap}_{2,j}} + \frac{\delta b_F}{\mathrm{BoundRelGap}_{1,j}} \right)$$

It holds,

$$RHS_{\mathrm{RelGap}} \leq RHS_{\mathrm{BoundRelGap}}.$$

$$\text{RelGap}_{1,j} \geq \min_{i\neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{|\widetilde{\widetilde{\lambda}}_i(\mathbf{v})| + M_{i_1}} =: \text{BoundRelGap}_{1,j},$$

$$\text{RelGap}_{2,j} \geq \min_{i\neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{\lambda_j(\mathbf{v^0})} =: \text{BoundRelGap}_{2,j},$$

where $M_{i_1} = \frac{1}{2} M_i \|(\mathbf{v} - \mathbf{v}^0)\|^2$.

$$RHS_{\text{BoundRelGap}} = \kappa(X)\kappa(\widetilde{X}) \left(\frac{\delta a_F + \delta c_F}{\text{BoundRelGap}_{2,j}} + \frac{\delta b_F}{\text{BoundRelGap}_{1,j}} \right)$$

It holds,

$$RHS_{\text{RelGap}} \leq RHS_{\text{BoundRelGap}}.$$

$$\mathrm{RelGap}_{1,j} \geq \min_{i \neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{|\widetilde{\widetilde{\lambda}}_i(\mathbf{v})| + M_{i_1}} =: \mathrm{BoundRelGap}_{1,j},$$

$$\mathrm{RelGap}_{2,j} \geq \min_{i \neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{\lambda_j(\mathbf{v^0})} =: \mathrm{BoundRelGap}_{2,j},$$

where $M_{i_1} = \frac{1}{2} M_i \|(\mathbf{v} - \mathbf{v}^0)\|^2$.

$$RHS_{\mathrm{BoundRelGap}} = \kappa(X)\kappa(\widetilde{X}) \left(\frac{\delta a_F + \delta c_F}{\mathrm{BoundRelGap}_{2,j}} + \frac{\delta b_F}{\mathrm{BoundRelGap}_{1,j}} \right)$$

It holds,

$$RHS_{\mathrm{RelGap}} \leq RHS_{\mathrm{BoundRelGap}}.$$

$$\mathrm{RelGap}_{1,j} \geq \min_{i\neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{|\widetilde{\widetilde{\lambda}}_i(\mathbf{v})| + M_{i_1}} =: \mathrm{BoundRelGap}_{1,j},$$

$$\mathrm{RelGap}_{2,j} \geq \min_{i\neq j} \frac{|\lambda_j(\mathbf{v^0}) - \widetilde{\widetilde{\lambda}}_i(\mathbf{v})| - M_{i_1}}{\lambda_j(\mathbf{v^0})} =: \mathrm{BoundRelGap}_{2,j},$$

where $M_{i_1} = \frac{1}{2} M_i \|(\mathbf{v} - \mathbf{v}^0)\|^2$.

$$RHS_{\mathrm{BoundRelGap}} = \kappa(X)\kappa(\widetilde{X}) \left(\frac{\delta a_F + \delta c_F}{\mathrm{BoundRelGap}_{2,j}} + \frac{\delta b_F}{\mathrm{BoundRelGap}_{1,j}} \right)$$

It holds,

$$RHS_{\mathrm{RelGap}} \leq RHS_{\mathrm{BoundRelGap}}.$$

Numerical example

Example 2 Let dimension be $n = 100$, and M and K are the same as in Example 1.

$D(\mathbf{v}^0) = \alpha C_{crit} + C_{ext}(\mathbf{v}^0)$, where $\alpha = 0.004$,

$$C_{ext}(\mathbf{v}^0) = \mathrm{diag}(v_1^0 I_{40}, v_2^0 I_{50}, v_3^0 I_{10}).$$

Parameters $\mathbf{v}$ and $\mathbf{v^0}$ are as follows

$$\mathbf{v} = (c + \delta, \tfrac{c}{2} + \delta, 4c + \delta, \delta) \text{ and } \mathbf{v^0} = (c, \tfrac{c}{2}, 4c, 0),$$

where $c = 1000$ and $\delta = 0.00025$.

Relative error of right hand side is as defined:

$$Rel_{err} = \frac{|RHS_{\mathrm{RelGap}} - RHS_{\mathrm{BoundRelGap}}|}{RHS_{\mathrm{RelGap}}}.$$

Numerical example

Example 2 Let dimension be $n = 100$, and M and K are the same as in Example 1.
$D(\mathbf{v}^0) = \alpha C_{crit} + C_{ext}(\mathbf{v}^0)$, where $\alpha = 0.004$,

$$C_{ext}(\mathbf{v}^0) = \mathrm{diag}(v_1^0 I_{40}, v_2^0 I_{50}, v_3^0 I_{10}).$$

Parameters $\mathbf{v}$ and $\mathbf{v^0}$ are as follows

$$\mathbf{v} = (c + \delta, \tfrac{c}{2} + \delta, 4c + \delta, \delta) \text{ and } \mathbf{v^0} = (c, \tfrac{c}{2}, 4c, 0),$$

where $c = 1000$ and $\delta = 0.00025$.
Relative error of right hand side is as defined:

$$Rel_{err} = \frac{|RHS_{\mathrm{RelGap}} - RHS_{\mathrm{BoundRelGap}}|}{RHS_{\mathrm{RelGap}}}.$$

Numerical example

Example 2 Let dimension be $n = 100$, and M and K are the same as in Example 1.
$D(\mathbf{v}^0) = \alpha C_{crit} + C_{ext}(\mathbf{v}^0)$, where $\alpha = 0.004$,

$$C_{ext}(\mathbf{v}^0) = \mathrm{diag}(v_1^0 I_{40}, v_2^0 I_{50}, v_3^0 I_{10}).$$

Parameters $\mathbf{v}$ and $\mathbf{v^0}$ are as follows

$$\mathbf{v} = (c+\delta, \tfrac{c}{2}+\delta, 4c+\delta, \delta) \text{ and } \mathbf{v^0} = (c, \tfrac{c}{2}, 4c, 0),$$

where $c = 1000$ and $\delta = 0.00025$.
Relative error of right hand side is as defined:

$$Rel_{err} = \frac{|RHS_{\mathrm{RelGap}} - RHS_{\mathrm{BoundRelGap}}|}{RHS_{\mathrm{RelGap}}}.$$

Numerical example

Example 2 Let dimension be $n = 100$, and M and K are the same as in Example 1.
$D(\mathbf{v}^0) = \alpha C_{crit} + C_{ext}(\mathbf{v}^0)$, where $\alpha = 0.004$,

$$C_{ext}(\mathbf{v}^0) = \mathrm{diag}(v_1^0 I_{40}, v_2^0 I_{50}, v_3^0 I_{10}).$$

Parameters $\mathbf{v}$ and $\mathbf{v^0}$ are as follows

$$\mathbf{v} = (c + \delta, \tfrac{c}{2} + \delta, 4c + \delta, \delta) \text{ and } \mathbf{v^0} = (c, \tfrac{c}{2}, 4c, 0),$$

where $c = 1000$ and $\delta = 0.00025$.
Relative error of right hand side is as defined:

$$Rel_{err} = \frac{|RHS_{\mathrm{RelGap}} - RHS_{\mathrm{BoundRelGap}}|}{RHS_{\mathrm{RelGap}}}.$$

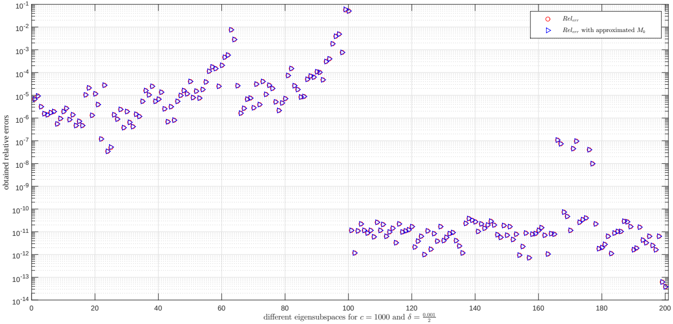


Figure: Relative error of right hand side

Thank you for your attention!