Greedy optimal control for elliptic equations

Applications to turnpike control

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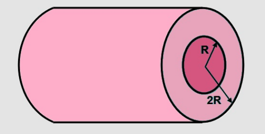
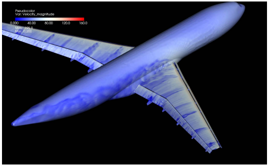
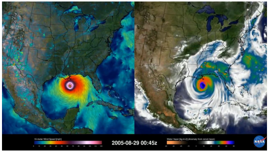
Parameter dependent problems

Real life applications (may) depend on a large number of parameters



examples: thickness, conductivity, density, length, humidity, pressure, curvature,. . .





Parameter dependent problems (Cont.)

- When dealing with applications and simulations, we would like to explore within different parameter configurations.
- From the control point of view, this implies solving a different problem for each configuration.
- Computationally expensive.

Our goal

Apply greedy theory to have a robust and fast numerical solvers.

Parameter dependent control problem

$\Omega \subset \mathbb{R}^N, \quad \omega \subset \Omega.$

Consider the system

$$\begin{cases} -\text{div}(a(x,\nu)\nabla y) + c\,y = \chi_\omega u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

∘ ν is a parameter ∘ $u \in L^2(\omega)$ is a control ∘ $c = c(x) \in L^\infty(\Omega)$

Optimal control problem (OCP$_\nu$)

$$\min_{u \in L^2(\omega)} J_\nu(u) = \frac{1}{2}|u|^2_{L^2(\omega)} + \frac{\beta}{2}\|y - y_d\|^2_{L^2(\Omega)},$$

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∃! optimal solution is well-known (Lions, Tröltzsch,. . .)

Characterization: optimal pair $(\bar{u}, \bar{y})$

$$\bar{u} = -\chi_\omega \bar{q}$$

$$\begin{cases} -\text{div}(a(x,\nu)\nabla \bar{y}) + c\,\bar{y} = -\chi_\omega \bar{q}, & \text{in } \Omega, \\ -\text{div}(a(x,\nu)\nabla \bar{q}) + c\,\bar{q} = \beta\,(\bar{y} - y^d), & \text{in } \Omega, \\ \bar{y} = \bar{q} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

As the state y depends on ν, also the control u depends on ν.

Parameter dependent control problem (cont.)

$$\begin{cases} -\text{div}(a(x,\nu)\nabla y) + c\,y = \chi_\omega u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases}$$

From the practical point of view,

- Measure parameter ν and determine u_ν

$$\min_{u \in L^2(\omega)} J_\nu(u) = \frac{1}{2}|u|^2_{L^2(\omega)} + \frac{\beta}{2}\|y - y_d\|^2_{L^2(\Omega)},$$

  using classical methods (iterative methods, ...)

- Repeat the process for each new value of ν.

Can we do it better?

Assume that ν ranges within a compact set $\mathcal{K} \subset \mathbb{R}^d$ and $a_\nu = a(x, \nu)$ are bounded functions satysfing

$$0 < a_1 \leq a_\nu \leq a_2, \qquad \nu \in K.$$

In this way, we ensure that each control can be uniquely determined by

$$\bar{u}_\nu = -\chi_\omega \bar{q}$$

where $(\bar{y}, \bar{q})$ solve the optimality system (14). Consider the set of controls $\bar{u}_\nu$ for each possible value $\nu \in \mathcal{K}$. That is,

$$\bar{\mathcal{U}} = \{\bar{u}_\nu : \nu \in \mathcal{K}\}$$

The idea

To determine a finite number of values of ν that yield the best possible approximation of the control manifold $\bar{\mathcal{U}}$

Description of the method

We look for a *small* number of parameters $\nu \in \mathcal{K}$ approximating the manifold $\bar{\mathcal{U}}$ in the sense of the Kolmogorov width. **Roughly**, the Kolmogorov width measures how well we can approximate $\bar{\mathcal{U}}$ by a finite dimensional space.

In order to achieve this goal we rely on greedy algorithms and reduced bases methods for parameter dependent PDEs or abstract equations in Banach spaces.

A. Cohen, R. DeVore, Kolmogorov widths under holomorphic mappings, *IMA Journal on Numerical Analysis*, to appear

A. Cohen, R. DeVore, Approximation of high-dimensional parametric PDEs, arXiv preprint, 2015.

Y. Maday, O. Mula, A. T. Patera, M. Yano, The generalized Empirical Interpolation Method: stability theory on Hilbert spaces with an application to the Stokes equation, submitted

The pure greedy method

X – a Banach space $K \subset X$ – a compact subset.

The method approximates K by a a series of finite dimensional linear spaces V_n (a linear method).

The algorithm

The first step Choose $x_1 \in K$ such that

$$\|x_1\|_X = \max_{x \in K} \|x\|_X.$$

The general step Having found $x_1..x_n$, denote $V_n = \operatorname{span}\{x_1, \ldots, x_n\}$. Choose the next element

$$x_{n+1} := \arg\max_{x \in K} \operatorname{dist}(x, V_n). \tag{3}$$

The algorithm stops when $\sigma_n(K) := \max_{x \in K} \operatorname{dist}(x, V_n)$ becomes less than the given tolerance ε.

The greedy idea

The greedy idea

Which one you are going to choose?

Sometimes it is hard to solve the maximisation problem (3).

The weak greedy method

– a relaxed version of the pure one.

The algorithm

Fix a constant $\gamma \in \langle 0, 1]$.

The first step Choose $x_1 \in K$ such that

$$\|x_1\|_X \geq \gamma \max_{x \in K} \|x\|_X .$$

The general step

Having found $x_1..x_n$, denote $V_n = \mathrm{span}\{x_1, \dots, x_n\}$.

Choose the next element

$$\mathrm{dist}(x_{n+1}, V_n) \geq \gamma \max_{x \in K} \mathrm{dist}(x, V_n) . \tag{4}$$

The algorithm stops when $\sigma_n(K) := \max_{x \in K} \mathrm{dist}(x, V_n)$ becomes less than the given tolerance ε.

Efficiency

In order to estimate **the efficiency of the (weak) greedy algorithm** we compare its approximation rates $\sigma_n(K)$ with the best possible one.

The Kolmogorov n width, $d_n(K)$

– measures how well K can be approximated by a subspace in X of a fixed dimension n.

$$d_n(K) := \inf_{\dim Y = n} \sup_{x \in K} \inf_{y \in Y} \|x - y\|_X \, .$$

Thus $d_n(K)$ represents optimal approximation performance that can be obtained by a n-dimensional linear space.

The greedy approximation rates have same decay as the Kolmogorov widths.

Performance obstacles

- The set K in general consists of infinitely many vectors.
- In practical implementations the set K is often unknown (e.g. it represents the family of solutions to parameter dependent problems).

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PRACTICAL REALISATION DEPENDS CRUCIALLY ON AN EXISTENCE OF AN APPROPRIATE SURROGATE .

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Los Alamos National Laboratory

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The surrogate

In practical implementations, the set $\bar{\mathcal{U}}$ is unknown.

Given two parameters ν_1 and ν_2, how can we measure the distance between $\bar{u}_{\nu_1}$ and $\bar{u}_{\nu_2}$?

Recall that we want to avoid to compute $\bar{u}_\nu$.

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Standard residual: Suppose that we have computed u_{ν_1}

$$|u_{\nu_1} - u_{\nu_2}| \sim |\nabla J_{\nu_2}(u_{\nu_1}) - \nabla J_{\nu_2}(u_{\nu_2})| = \nabla J_{\nu_2}(u_{\nu_1})$$

Compute $\nabla J_{\nu_2}(u_{\nu_1}) = u_{\nu_1} + \beta S^*_{\nu_2}(S_{\nu_2} u_{\nu_1} - y_d)$, where S_ν is to control-to-state operator. This means

$$\begin{cases} -\text{div}(a_{\nu_2} \nabla y) + c\, y = \chi_\omega u_{\nu_1}, & \text{in } \Omega, \\ -\text{div}(a_{\nu_2} \nabla q) + c\, q = \beta\, (y - y^d), & \text{in } \Omega, \\ y = q = 0, & \text{on } \partial\Omega. \end{cases} \quad \Rightarrow -\chi_\omega q = S^*_{\nu_2}(S_{\nu_2} u_{\nu_1} - y_d)$$

Solving a cascade system.

Cheaper surrogates

A cheap surrogate: Instead of using $\bar{u}_\nu$ and approximate the manifold $\bar{\mathcal{U}}$, use the optimal variables $(\bar{q}_\nu, \bar{y}_\nu)$ and approximate the manifold $\bar{\mathcal{Q}} \times \bar{\mathcal{Y}}$.

Denoting $L_\nu z := -\text{div}(a_\nu \nabla z) + c\, z$, we define

$$R_\nu(q, y) := \begin{pmatrix} L_\nu y + \chi_\omega q \\ L_\nu q - \beta\,(y - y_d) \end{pmatrix} = \underbrace{G_\nu(q, y)}_{\text{linear part of} R_\nu(q,y)} + \begin{pmatrix} 0 \\ \beta y_d \end{pmatrix}.$$

With this definition, we are able to compute the following estimates:

$$c_1 \left(\|y - \bar{y}_\nu\|_{H_0^1 \Omega} + \|q - \bar{q}_\nu\|_{H_0^1(\Omega)} \right) \leq \|R_\nu(q, y)\|_{H^{-1}(\Omega)},$$

$$\|R_\nu(p, y)\|_{H^{-1}\Omega} \leq (1 + \alpha_2)(\|y - \bar{y}_\nu\|_{H_0^1(\Omega)} + \|q - \bar{q}_\nu\|_{H_0^1(\Omega)}).$$

where c_1 and α_2 only depending on a_1, a_2 and $\|c\|_\infty$.

Upper and lower bounds for $R_\nu(q, y)$ are essential for the proof of greedy algorithms in terms of the Kolmogorov width.

Main results

$$R_\nu(q,y) := \begin{pmatrix} L_\nu y + \chi_\omega q \\ L_\nu q - \beta\,(y - y_d) \end{pmatrix}. \tag{5}$$

Theorem 1 (H. Santmaria, L., Zuazua, '17)

The residual (5) provides the approximation estimates for optimal controls and states

- $\|u_\nu^\star - \bar{u}_\nu\|_{L^2(\Omega)} \leq \dfrac{1}{c_1}\|R_\nu(q,y)\|_{[H^{-1}(\Omega)]^2}$,
- $\|y_\nu^\star - \bar{y}_\nu\|_{H_0^1(\Omega)} \leq \left(\dfrac{1}{\alpha_1 c_1}\right)\|R_\nu(q,y)\|_{[H^{-1}(\Omega)]^2}$,

where c_1 and α_1 only depend on a_1, a_2 and $\|c\|_\infty$.

Offline algorithm

Step 1: *Initialization*. Fix $\varepsilon > 0$. Choose any $\nu \in \mathcal{K}$, $\nu = \nu_1$ and compute the minimizer of J_{ν_1}. This leads to $\begin{pmatrix} \bar{q}_{\nu_1} \\ \bar{y}_{\nu_1} \end{pmatrix}$.

Step 2: *recursive choice of ν*.

Assuming we have chosen $\nu_1, \ldots, \nu_p$, we choose ν_{p+1} as the maximizer of

$$\max_{\nu \in \mathcal{K}} \| \inf_{(q,y) \in (\bar{\mathcal{Q}}_p, \bar{\mathcal{Y}}_p)} R_\nu(q, y) \|$$

Step 3: *Stopping criterion*. Stop if the $\max \leq \varepsilon$.

Theorem 1 (H-Santamaria, L., Zuazua, '17)

The offline algorithm stops after $n_0(\varepsilon)$ iterations, and fullfills the requirements of the greedy theory.

Online algorithm

After choosing the most representative values of ν, we can construct an approximated optimal control $u_\nu^\star$ for any arbitrary given value $\nu \in \mathcal{K}$ by taking

$$u_{\nu}^\star = \sum_{i=1}^{k} \lambda_i \bar{q}_{\nu_i}|_\omega$$

where λ_i are determined by the projection of the vector $\begin{pmatrix} 0 \\ y_d \end{pmatrix}$ to the space

$$\text{span}\{G_\nu(\bar{q}_{\nu_1}, \bar{y}_{\nu_1}), \ldots, G_\nu(\bar{q}_{\nu_k}, \bar{y}_{\nu_k})\}$$

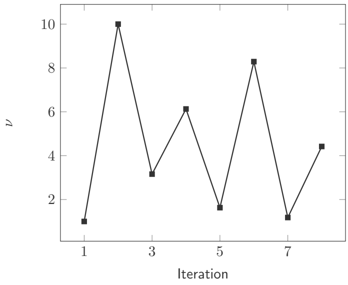
Numerical results

Numerical examples

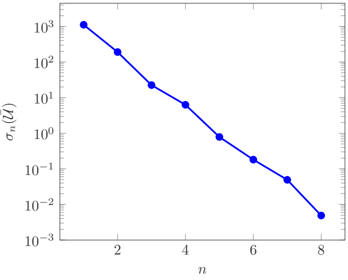
- $\Omega = (0,1)^2$ in 2-D or $\Omega = (0,1)$ in 1-D.
- Uniform meshes, i.e., meshes with constant discretization steps in each direction, $N = 400$.
- We will approximate the operator $\mathcal{A} = -\text{div}(a(x,\nu)\nabla\cdot)$ by using the standard 5-point discretization.
- Discretize-then-optimize.
- $\nu \in \mathcal{K} = [1, 10]$.
- $\mathcal{K}$ sampled in 100 equidistant points.

Greedy test # 1

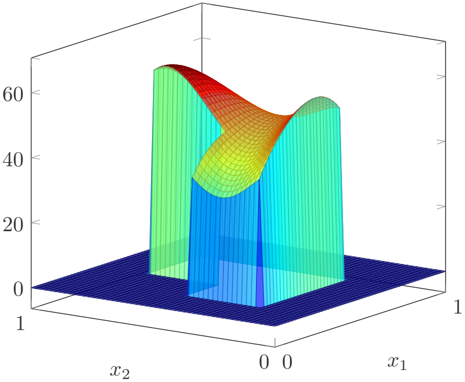
- $a(x, \nu) = 1 + \nu(x_1^2 + x_2^2)$, ∘ $c(x) = \sin(2\pi x_1)\sin(2\pi x_2)$,
- $y_d = sin(\pi x_1)$, ∘ $\beta = 10^4$, ∘ $\varepsilon = 0.005$
- $t_{\text{cheap}} = 304s$, ∘ $t_{std} = 384s$



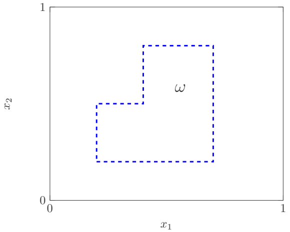
(a) Selected ν



(b) Approximation error



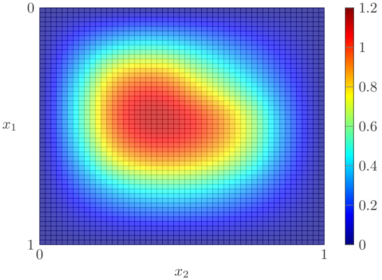
(c) The approximated control



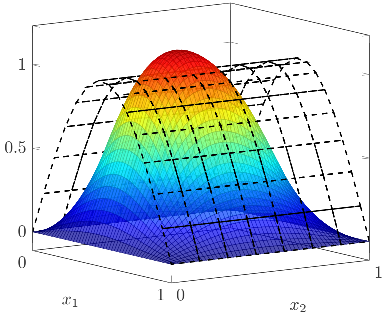
(d) The control set ω

$\circ\, |u^{\star}_{\pi/2} - \bar{u}_{\pi/2}|_{L^2(\omega)} \approx 1.45 \times 10^{-5}, \quad t_{\text{online}} = 0.45s, \quad t_{\text{iterative}} = 6.01s.$

Approximation for $\nu = \pi/2$ (cont.)



(e) The controlled state



(f) The state $y^{\star}_{\pi/2}$ and the target function y^d (dashed)

$\circ\ |y^{\star}_{\pi/2} - \bar{y}_{\pi/2}|_{L^2(\Omega)} \approx 1.15 \times 10^{-7}$

Connection with the turnpike problems

Time dependent control problem

Consider

$$\begin{cases} \partial_t y - \text{div}(a(x,\nu)\nabla y) + c\,y = \chi_\omega u & \text{in } Q = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(x,0) = y^0(x) & \text{in } \Omega. \end{cases} \quad (6)$$

and the control problem

$$\min_u J_\nu{}^T(u) = \frac{1}{2}\int_0^T |u(t)|^2_{L^2(\omega)} dt + \frac{\beta}{2}\int_0^T \|y(t) - y^d\|^2_{L^2(\Omega)} dt.$$

The optimal solution (u^T, y^T) satisfies

$$\|y^T(t) - \bar{y}\|_{L^2(\Omega)} + \|u^T(t) - \bar{u}\|_{L^2(\Omega)} \leq K\left(e^{-\mu t} + e^{-\mu(T-t)}\right), \quad \forall t \in [0,T]$$

- Exponential convergence of the finite-time horizon control problem to the steady one as $T \to \infty$.

Greedy test # 2

We consider time-dependent version of the last example.

$\circ\, a(x,\nu) = 1 + \nu(x_1^2 + x_2^2)$, $\quad\circ\, c(x) = \sin(2\pi x_1)\sin(2\pi x_2)$,
$\circ\, y_d = sin(\pi x_1)$, $\quad\circ\, \beta = 10^4$, $\quad\circ\, \varepsilon = 0.005$
$\circ\Omega = (0,1)^2$ in 2-D or $\Omega = (0,1)$ in 1-D.

We take initial datum
$y_0(x) = \sin(3\pi x_1)\sin(2\pi x_2)$

The case $c(x) \geq 0$ (greedy test #2)

$$u(x,t) = u^{\star}_{\pi/2}(x), \qquad y_0(x) = \sin(3\pi x_1)\sin(2\pi x_2)$$