Primjena homogenizacije na problem optimalnog dizajna uz pretpostavku malih amplituda

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- "Optimal Design in Small Amplitude Homogenization" G. Allaire, S. Gutiérrez,2005.
- "Shape Optimization by the Homogenization Method" G. Allaire, Springer, 2010.
- promatramo mješavinu dva materijala A^0 i A^1 u kojem je A^1 mala smetnja (pertubacija) A^0
- A^0 i A^1 simetrični pozitivno definitni tenzori
- η amplituda ili kontrast između dva materijala, tj. $A^1 = A^0(1+\eta)$
- s χ ćemo označavati karakterističnu funkciju područja okupiranog materijalom A^1
- tenzor provođenja (conductivity tensor)

$$A(x) = (1-\chi(x))A^0 + \chi(x)A^1 = A^0(1+\eta\chi(x))$$

- neka je $\Omega \subset \mathbb{R}^N$ gladak, omeđen, otvoren skup, s rubom $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$
- za dane $f \in H^{-1}(\Omega)$ i $g \in L^2(\partial\Omega)$, promatramo problem:

$$\begin{cases} -\text{div}(A\nabla u^0) = f \text{ u } \Omega \\ u = 0 \text{ na } \Gamma_D \\ A\nabla u^0 \cdot n = g \text{ na } \Gamma_N \end{cases} \tag{1}$$

- želimo minimizirati: $J(\chi) = \int_\Omega j_1(u)dx + \int_{\Gamma_N} j_2(u)ds$, $j_1, j_2 \in C^3$ s adekvatnim uvjetom rasta:
npr. $\exists C > 0\ |j_i(u)| \leq C(|u|^2 + 1)$, $|j_i'(u)| \leq C(|u| + 1)$, $|j_i''(u)| \leq C$.
- Primjeri funkcije cilja:

$$J(\chi) = \int_\Omega f(x)u(x)dx \text{ ili } J(\chi) = \int_\Omega |u(x) - u_{cilj}(x)|^2 dx\,.$$

- uz pretpostavku da materijali imaju zadani omjer volumena, Θ za A^1 i $(1-\Theta)$ za A^0, $\Theta \in (0,1)$, definiramo skup dopustivih dizajna s:

$$\mathcal{U}_{ad} = \left\{ \chi \in L^\infty(\Omega; \{0,1\}) : \int_\Omega \chi(x)dx = \Theta|\Omega| \right\} . \qquad (2)$$

Definicija 1

Optimizacijski problem

$$\inf_{\chi \in \mathcal{U}_{ad}} J(\chi) \qquad (3)$$

zovemo **problem optimalnog dizajna uz velike amplitude***.*

Problem za kojeg ne postoji optimalan dizajn

Neka je u rješenje zadaće:

$$\begin{cases} -\text{div}(A\nabla u) = 0 \text{ u } \Omega \\ A\nabla u \cdot n = \sigma_0 \cdot n \text{ na } \partial\Omega \end{cases},$$

gdje je $\sigma_0 = |\sigma_0| e_N$ i $A(x) = \alpha \mathrm{I}\chi(x) + \beta \mathrm{I}(1-\chi(x))$.
Funkcija cilja je:

$$J(\chi) = \int_{\partial\Omega} (\sigma_0 \cdot n) u ds + l \int_{\Omega} (1 - \chi(x)) dx\,,$$

gdje je $\frac{|\sigma_0|^2(\beta-\alpha)}{\beta^2} < l < \frac{|\sigma_0|^2(\beta-\alpha)}{\alpha^2}$.
Problem: minimizirati J u prostoru $L^\infty(\Omega; \{0,1\})$.

Pretpostavka malih amplituda

Kako je A linearna funkcija u η, rješenje $u \in H^1(\Omega)$ je analitičko u odnosu na η:

$$u = u^0 + \eta u^1 + \eta^2 u^2 + \dots$$

Uvrštavanjem u problem (1):

$$\begin{aligned}
f &= -\mathrm{div}(A\nabla u) \\
&\approx -\mathrm{div}\left(A^0(1+\eta\chi)(\nabla u^0 + \eta\nabla u^1 + \eta^2\nabla u^2)\right) \\
&= -\mathrm{div}\left(A^0\nabla u^0 + \eta(A^0\nabla u^1 + A^0\chi\nabla u^0) + \right. \\
&\quad \left. \eta^2\left(A^0\nabla u^2 + A^0\chi\nabla u^1\right)\right)
\end{aligned}$$

Iz toga dobivamo:

$$\begin{cases} -\mathrm{div}(A^0 \nabla u^0) = f \text{ u } \Omega \\ u^0 = 0 \text{ na } \Gamma_D \\ A^0 \nabla u^0 \cdot n = g \text{ na } \Gamma_N \end{cases} \tag{4}$$

$$\begin{cases} -\mathrm{div}(A^0 \nabla u^1) = \mathrm{div}(\chi A^0 \nabla u^0) \text{ u } \Omega \\ u^1 = 0 \text{ na } \Gamma_D \\ A^0 \nabla u^1 \cdot n = -\chi A^0 \nabla u^0 \cdot n \text{ na } \Gamma_N \end{cases} \tag{5}$$

$$\begin{cases} -\mathrm{div}(A^0 \nabla u^2) = \mathrm{div}(\chi A^0 \nabla u^1) \text{ u } \Omega \\ u^2 = 0 \text{ na } \Gamma_D \\ A^0 \nabla u^2 \cdot n = -\chi A^0 \nabla u^1 \cdot n \text{ na } \Gamma_N \end{cases} \tag{6}$$

Taylorov razvoj za $J(\chi) = \int_\Omega j_1(u)dx + \int_{\Gamma_N} j_2(u)ds$:

$$\begin{aligned} j_i(u) &= j_i(u^0) + j_i'(u^0)(u - u^0) + \frac{j_i''(u^0)}{2}(u - u^0)^2 + \dots \\ &\approx j_i(u^0) + j_i'(u^0)(u^0 + \eta u^1 + \eta^2 u^2 - u^0) + \\ &\quad \frac{j_i''(u^0)}{2}(u^0 + \eta u^1 + \eta^2 u^2 - u^0)^2 \\ &= j_i(u^0) + j_i'(u^0)(\eta u^1 + \eta^2 u^2) + \frac{j_i''(u^0)}{2}(\eta u^1 + \eta^2 u^2)^2 \end{aligned}$$

$$
\begin{aligned}
\mathcal{J}_{sa}(u^0, u^1, u^2) \;=\; & \int_\Omega j_1(u^0)dx + \eta \int_\Omega j_1'(u^0)u^1 dx + \\
& \eta^2 \int_\Omega \left(j_1'(u^0)u^2 + \frac{1}{2} j_1''(u^0)(u^1)^2 \right) dx + \\
& \int_{\Gamma_N} j_2(u^0)ds + \eta \int_{\Gamma_N} j_2'(u^0)u^1 ds + \\
& \eta^2 \int_{\Gamma_N} \left(j_2'(u^0)u^2 + \frac{1}{2} j_2''(u^0)(u^1)^2 \right) ds
\end{aligned}
$$

Definicija 2

Optimizacijski problem

$$
\inf_{\chi \in \mathcal{U}_{ad}} \{ J_{sa}(\chi) = \mathcal{J}_{sa}(u^0, u^1, u^2) \} \tag{7}
$$

gdje su u^0, u^1 i u^2 rješenja od (4), (5) i (6) zovemo **problem optimalnog dizajna uz male amplitude***.*

- Problem uz pretpostavku malih amplituda (7) je loše postavljen.
- Postupak za računanje relaksacije (7): promatramo niz (minimizirajućih ili ne) karakterističnih funkcija χ_n i prijeđemo na limes u (7) i pripadajućim jednadžbama stanja.
- Do na podniz, postoji limes θ takav da $\chi_n \overset{*}{\rightharpoonup} \theta$ (u $L^\infty(\Omega; [0,1])$). S u^0, u_n^1, u_n^2 označimo rješenja od (4) - (6) pridružena χ_n.

$$\begin{cases} -\mathrm{div}(A^0\nabla u^0) = f \text{ u } \Omega \\ u^0 = 0 \text{ na } \Gamma_D \\ A^0\nabla u^0 \cdot n = g \text{ na } \Gamma_N \end{cases}$$

$$\begin{cases} -\mathrm{div}(A^0\nabla u_n^1) = \mathrm{div}(\chi_n A^0\nabla u^0) \text{ u } \Omega \\ u_n^1 = 0 \text{ na } \Gamma_D \\ A^0\nabla u_n^1 \cdot n = -\chi_n A^0\nabla u^0 \cdot n \text{ na } \Gamma_N \end{cases}$$

$$\begin{cases} -\mathrm{div}(A^0\nabla u_n^2) = \mathrm{div}(\chi_n A^0\nabla u_n^1) \text{ u } \Omega \\ u_n^2 = 0 \text{ na } \Gamma_D \\ A^0\nabla u_n^2 \cdot n = -\chi_n A^0\nabla_n u^1 \cdot n \text{ na } \Gamma_N \end{cases}$$

$$\begin{cases} -\mathrm{div}(A^0 \nabla u_n^1) = \mathrm{div}(\chi_n A^0 \nabla u^0) \text{ u } \Omega \\ u_n^1 = 0 \text{ na } \Gamma_D \\ A^0 \nabla u_n^1 \cdot n = -\chi_n A^0 \nabla u^0 \cdot n \text{ na } \Gamma_N \end{cases}$$

$$\int_\Omega A^0 \nabla u_n^1 \cdot \nabla \phi \;=\; \int_\Omega \chi_n A^0 \nabla u^0 \cdot \nabla \phi$$

$$\begin{cases} -\mathrm{div}(A^0\nabla u_n^1) = \mathrm{div}(\chi_n A^0\nabla u^0) \text{ u } \Omega \\ u_n^1 = 0 \text{ na } \Gamma_D \\ A^0\nabla u_n^1 \cdot n = -\chi_n A^0\nabla u^0 \cdot n \text{ na } \Gamma_N \end{cases}$$

$$\begin{array}{ccc} \int_\Omega A^0\nabla u_n^1 \cdot \nabla\phi & = & \int_\Omega \chi_n A^0\nabla u^0 \cdot \nabla\phi \\ \downarrow & & \downarrow \end{array}$$

$$\begin{cases} -\mathrm{div}(A^0\nabla u_n^1) = \mathrm{div}(\chi_n A^0 \nabla u^0) \text{ u } \Omega \\ u_n^1 = 0 \text{ na } \Gamma_D \\ A^0\nabla u_n^1 \cdot n = -\chi_n A^0\nabla u^0 \cdot n \text{ na } \Gamma_N \end{cases}$$

$$\begin{array}{ccc} \int_\Omega A^0\nabla u_n^1 \cdot \nabla\phi & = & \int_\Omega \chi_n A^0\nabla u^0 \cdot \nabla\phi \\ \downarrow & & \downarrow \\ \int_\Omega A^0\nabla u^1 \cdot \nabla\phi & = & \int_\Omega \theta A^0\nabla u^0 \cdot \nabla\phi \end{array}$$

$$\begin{cases} -\mathrm{div}(A^0\nabla u^1) = \mathrm{div}(\theta A^0 \nabla u^0) \text{ u } \Omega \\ u^1 = 0 \text{ na } \Gamma_D \\ A^0\nabla u^1 \cdot n = -\theta A^0\nabla u^0 \cdot n \text{ na } \Gamma_N \end{cases} \tag{8}$$

$$\begin{cases} -\mathrm{div}(A^0 \nabla u_n^2) = \mathrm{div}(\chi_n A^0 \nabla u_n^1) \text{ in } \Omega \\ u_n^2 = 0 \text{ na } \Gamma_D \\ A^0 \nabla u_n^2 \cdot n = -\chi_n A^0 \nabla_n u^1 \cdot n \text{ na } \Gamma_N \end{cases}$$

$$\int_\Omega A^0 \nabla u_n^2 \cdot \nabla \phi \;=\; \int_\Omega \chi_n A^0 \nabla u_n^1 \cdot \nabla \phi$$

$$\begin{cases} -\mathrm{div}(A^0 \nabla u_n^2) = \mathrm{div}(\chi_n A^0 \nabla u_n^1) \text{ in } \Omega \\ u_n^2 = 0 \text{ na } \Gamma_D \\ A^0 \nabla u_n^2 \cdot n = -\chi_n A^0 \nabla_n u^1 \cdot n \text{ na } \Gamma_N \end{cases}$$

$$\int_\Omega A^0 \nabla u_n^2 \cdot \nabla \phi \quad = \quad \int_\Omega \chi_n A^0 \nabla u_n^1 \cdot \nabla \phi$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{cases} -\mathrm{div}(A^0\nabla u_n^2) = \mathrm{div}(\chi_n A^0 \nabla u_n^1) \text{ in } \Omega \\ u_n^2 = 0 \text{ na } \Gamma_D \\ A^0 \nabla u_n^2 \cdot n = -\chi_n A^0 \nabla_n u^1 \cdot n \text{ na } \Gamma_N \end{cases}$$

$$\begin{array}{ccc} \int_\Omega A^0 \nabla u_n^2 \cdot \nabla \phi & = & \int_\Omega \chi_n A^0 \nabla u_n^1 \cdot \nabla \phi \\ \downarrow & & \downarrow \\ \int_\Omega A^0 \nabla u^2 \cdot \nabla \phi & = & ??? \end{array}$$

Trebamo naći koju jednadžbu zadovoljava u^2.
Iz (5) vidimo da ∇u_n^1 ovisi linearno o χ_n, kroz pseudo-diferencijalni operator q, reda 0, čiji je simbol

$$q(x,\xi) = -\frac{A^0 \nabla u^0(x) \cdot \xi}{A^0 \xi \cdot \xi} \xi \,.$$

H-mjera podniza karakterističnih funkcija χ_n je oblika:

$$\mu(dx, d\xi) = \theta(x)(1-\theta(x))\nu(dx, d\xi)\,,$$

gdje je ν vjerojatnosna mjera u odnosu na ξ.

$$\lim_{n\to\infty} \int_\Omega \chi_n A^0 \nabla u_n^1 \cdot \nabla \phi dx = \int_\Omega \theta A^0 \nabla u^1 \cdot \nabla \phi dx -$$
$$\int_\Omega \int_{\mathbb{S}_{N-1}} \theta(1-\theta) \frac{A^0 \nabla u^0 \cdot \xi}{A^0 \xi \cdot \xi} \xi A^0 \nabla \phi \nu(dx, d\xi).$$

Uz $M(x) = \int_{\mathbb{S}_{N-1}} \frac{\xi \otimes \xi}{A^0\xi \cdot \xi} \nu(x, d\xi)$ dobijemo:

$$\int_\Omega A^0 \nabla u^2 \cdot \nabla \phi dx = -\int_\Omega \theta A^0 \nabla u^1 \cdot \nabla \phi dx + \int_\Omega \theta(1-\theta) A^0 M A^0 \nabla u^0 \cdot \nabla \phi dx$$

za bilo koju glatku test funkciju ϕ koja iščezava na Γ_D.
Stoga u_2 je jedinstveno rješenje u $H^1(\Omega)$ od

$$\begin{cases} -\mathrm{div}(A^0 \nabla u^2) = \mathrm{div}(\theta A^0 \nabla u^1) - \mathrm{div}(\theta(1-\theta) A^0 M A^0 \nabla u^0) \text{ u } \Omega \\ u^2 = 0 \text{ na } \Gamma_D \\ A^0 \nabla u^2 \cdot n = -\theta A^0 \nabla u^1 \cdot n + \theta(1-\theta) A^0 M A^0 \nabla u^0 \cdot n \text{ na } \Gamma_N \end{cases} \tag{9}$$

Kako su ulaganja $H^1(\Omega)$ u $L^2(\Omega)$ i $L^2(\Gamma_N)$ kompaktna, lako dobijemo:

$$
\begin{aligned}
\lim_{n\to\infty} J_{sa}(\chi_n) &= \lim_{n\to\infty}\Big(\int_\Omega j_1(u^0)dx + \eta\int_\Omega j_1'(u^0)u_n^1 dx + \\
&\quad \eta^2\int_\Omega\Big(j_1'(u^0)u_n^2 + \frac{1}{2}j_1''(u^0)(u_n^1)^2\Big)\,dx + \\
&\quad \int_{\Gamma_N} j_2(u^0)ds + \eta\int_{\Gamma_N} j_2'(u^0)u_n^1 ds + \\
&\quad \eta^2\int_{\Gamma_N}\Big(j_2'(u^0)u_n^2 + \frac{1}{2}j_2''(u^0)(u_n^1)^2\Big)\,ds\Big) \\
&= J_{sa}^*(\theta,\nu) \\
&= \mathcal{J}_{sa}(u^0,u^1,u^2)\,,
\end{aligned}
$$

gdje su u^0, u^1 i u^2 rješenja zadaća (4), (8), (9).

Propozicija 1

Relaksacija problema (7) je stoga

$$\min_{(\theta,\nu)\in\mathcal{U}_{ad}^*} \{J_{sa}^*(\theta,\nu) = \mathcal{J}_{sa}(u^0,u^1,u^2)\} \tag{10}$$

gdje je $\mathcal{J}_{sa}(u^0,u^1,u^2)$ *funkcija cilja, a* u^0,u^1,u^2 *rješenja zadaća (4), (8), (9) i* $\mathcal{U}_{ad}^*$ *je definiran s*

$$\mathcal{U}_{ad}^* = \left\{ \begin{array}{c} (\theta,\nu) \in L^\infty(\Omega;[0,1]) \times \mathcal{P}(\Omega,\mathbb{S}_{N-1}) \\ t.d.\ \int_\Omega \theta(x)dx = \Theta|\Omega| \end{array} \right\}$$

gdje je skup vjerojatnosnih mjera $\mathcal{P}(\Omega,\mathbb{S}_{N-1})$ *definiran s*

$$\mathcal{P}(\Omega,\mathbb{S}_{N-1}) = \left\{ \begin{array}{c} \nu(x,\xi)\ \textit{Radonova mjera na}\ \Omega\times\mathbb{S}_{N-1}\ t.d. \\ \nu \ge 0, \int_{\mathbb{S}_{N-1}} \nu(x,\xi)d\xi = 1\ s.s\ x\in\Omega \end{array} \right\}.$$

Preciznije,

1. postoji barem jedan minimizator (θ, ν) problema (10)
2. bilo koji minimizator (θ, ν) problema (10) je dobiven pomoću minimizirajućeg niza χ_n problema (7) u smislu da $\chi_n \xrightarrow{*} \theta$ u $L^\infty(\Omega)$, $\theta(1-\theta)\nu$ je H-mjera niza $(\chi_n - \theta)$ i $\lim_{n\to\infty} J_{sa}(\chi_n) = J^*_{sa}(\theta, \nu)$
3. bilo koji minimizirajući niz χ_n problema (7) konvergira u prethodnom smislu k minimizatoru (θ, ν) problema (10).

Optimalni uvjeti

Propozicija 2

Relaksirani problem (10) može biti ekvivaletno riješen restrikcijom skupa vjerojatnosnih mjera $\mathcal{P}(\Omega; \mathbb{S}_{N-1})$ *na njegov podskup Diracovih masa. Preciznije, postoji optimalno dizajnersko rješenje problema*

$$\min_{(\theta,\nu)\in\mathcal{U}_{ad}^{sl}} J_{sa}^*(\theta,\nu) \tag{11}$$

gdje $\mathcal{U}_{ad}^{sl} \subset \mathcal{U}_{ad}^*$ *je definirano s*

$$\mathcal{U}_{ad}^{sl} = \left\{ \begin{array}{c} (\theta,\nu) \in L^\infty(\Omega; [0,1] \times \mathcal{P}(\Omega, \mathbb{S}_{N-1})) \ t.d. \\ \int_\Omega \theta(x)dx = \Theta|\Omega|, \nu(x,\xi) = \delta(\xi - \xi^*(x)) \ s.s \ x \in \Omega \end{array} \right\}.$$

Osim toga, optimalna Diracova masa H-mjere ne ovisi o θ.

Dokaz:

$$\begin{aligned} J^*_{sa}(\theta,\nu) &= \int_\Omega j_1(u^0)dx + \eta \int_\Omega j_1'(u^0)u^1 dx + \\ & \eta^2 \int_\Omega \left(j_1'(u^0)u^2 + \frac{1}{2} j_1''(u^0)(u^1)^2 \right) dx + \\ & \int_{\Gamma_N} j_2(u^0)ds + \eta \int_{\Gamma_N} j_2'(u^0)u^1 ds + \\ & \eta^2 \int_{\Gamma_N} \left(j_2'(u^0)u^2 + \frac{1}{2} j_2''(u^0)(u^1)^2 \right) ds \end{aligned}$$

Cilj je eliminirati u^2 iz $J^*_{sa}(\theta,\nu)$, pri tome koristimo p^0 rješenje u $H^1(\Omega)$ zadaće:

$$\left\{ \begin{array}{c} -\mathrm{div}(A^0 \nabla p^0) = j_1'(u^0) \text{ u } \Omega \\ p^0 = 0 \text{ na } \Gamma_D \\ A^0 \nabla p^0 \cdot n = j_2'(u^0) \text{ na } \Gamma_N \end{array} \right. \tag{12}$$

Kao i u^0, p^0 ne ovisi o (θ,ν).

Pomnožimo (12) s u^2 i parcijalno integriramo, učinimo isto za (9) množeći s p^0, te imamo

$$\int_\Omega j_1'(u^0)u^2 dx + \int_{\Gamma_N} j_2'(u_0)u^2 ds = -\int_\Omega \theta A^0 \nabla u^1 \cdot \nabla p^0 dx + \int_\Omega \theta(1-\theta)A^0 M A^0 \nabla u^0 \cdot \nabla p^0 dx$$

Preciznije, imamo

$$J^*_{sa}(\theta,\nu) = \int_\Omega j_1(u^0)dx + \eta \int_\Omega j_1'(u^0)u^1 dx + \eta^2 \int_\Omega \frac{1}{2} j_1''(u^0)(u^1)^2 dx - \eta^2 \int_\Omega \theta A^0 \nabla u^1 \cdot \nabla p^0 dx + \eta^2 \int_\Omega \theta(1-\theta)A^0 M A^0 \nabla u^0 \cdot \nabla p^0 dx + \int_{\Gamma_N} j_2(u^0)ds + \eta \int_{\Gamma_N} j_2'(u^0)u^1 ds + \eta^2 \int_{\Gamma_N} \frac{1}{2} j_2''(u^0)(u^1)^2 ds\,.$$

Pomnožimo (12) s u^2 i parcijalno integriramo, učinimo isto za (9) množeći s p^0, te imamo

$$\int_\Omega j_1'(u^0)u^2dx + \int_{\Gamma_N} j_2'(u_0)u^2ds = -\int_\Omega \theta A^0\nabla u^1\cdot\nabla p^0dx + \int_\Omega \theta(1-\theta)A^0MA^0\nabla u^0\cdot\nabla p^0dx$$

Preciznije, imamo

$$J_{sa}^*(\theta,\nu) = \int_\Omega j_1(u^0)dx + \eta\int_\Omega j_1'(u^0)u^1dx + \eta^2\int_\Omega \frac{1}{2}j_1''(u^0)(u^1)^2dx - \eta^2\int_\Omega \theta A^0\nabla u^1\cdot\nabla p^0dx + \eta^2\int_\Omega \theta(1-\theta)A^0MA^0\nabla u^0\cdot\nabla p^0dx + \int_{\Gamma_N} j_2(u^0)ds + \eta\int_{\Gamma_N} j_2'(u^0)u^1ds + \eta^2\int_{\Gamma_N}\frac{1}{2}j_2''(u^0)(u^1)^2ds\,.$$

Minimizirati $J^*_{sa}(\theta, \nu)$ u odnosu na ν znači minimizirati skalarnu, afinu funkciju na konveksnom skupu vjerojatnosnih mjera $\mathcal{P}(\Omega, \mathbb{S}_{N-1})$. Stoga bilo koji minimizator ν^* može biti zamijenjen drugim minimizatorom koji je Diracove mase koncentrirane u smjeru ξ^* koji minimizira funkciju

$$\frac{\xi \otimes \xi}{A^0 \xi \cdot \xi} A^0 \nabla u^0 \cdot A^0 \nabla p^0 \, .$$

ξ^* ne ovisi o θ.
Zamjena minimizatora ν^* s Diracovom masom koncentriranom u ξ^* ne mijenja θ, u^0, u^1 i p^0. Stoga možemo restringirati minimizaciju po ν na podskup od $\mathcal{P}(\Omega, \mathbb{S}_{N-1})$ Diracovih masa tipa

$$\nu(x, \xi) = \delta(\xi - \xi^*(x)) \, .$$

Napomena 1.

U slučaju kada je A^0 izotropna, možemo izračunati eksplicitno smjer ξ^* . Ako ili ∇u^0 ili ∇p^0 iščezavaju, svaki smjer je optimalan. U suprotnom, optimalan smjer je

$$\xi^* = \frac{e - e'}{\|e - e'\|} \text{ ako je } e \neq e', \xi^* \perp e \text{ ako je } e = e',$$

gdje $e = \dfrac{\nabla u^0}{|\nabla u^0|}$ *i* $e' = \dfrac{\nabla p^0}{|\nabla p^0|}$.

Nakon eliminacije mjere ν, tj. uvrštavanjem optimalne Diracove mase koncentrirane u $\xi^*(x)$, dobijemo da funkcija cilja ovisi samo o θ.

$$J_{sa}^*(\theta) = \int_\Omega j_1(u^0)dx + \eta \int_\Omega j_1'(u^0)u^1 dx + \eta^2 \int_\Omega \frac{1}{2} j_1''(u^0)(u^1)^2 dx -$$
$$\eta^2 \int_\Omega \theta A^0 \nabla u^1 \cdot \nabla p^0 dx + \eta^2 \int_\Omega \theta(1-\theta) A^0 M^* A^0 \nabla u^0 \cdot \nabla p^0 dx +$$
$$\int_{\Gamma_N} j_2(u^0) ds + \eta \int_{\Gamma_N} j_2'(u^0)u^1 ds + \eta^2 \int_{\Gamma_N} \frac{1}{2} j_2''(u^0)(u^1)^2 ds\,.$$

s $M^* = \dfrac{\xi^* \otimes \xi^*}{A^0 \xi^* \cdot \xi^*}$.

Možemo eliminirati u^1 u u članu 1.reda od η. Množenjem (12) s u^1 i parcijalnom integracijom, i činjenjem istog za (8) množenjem s p^0, dobijemo

$$\int_\Omega j_1'(u^0)u^1 dx + \int_{\Gamma_N} j_2'(u^0)u^1 ds = -\int_\Omega \theta A^0 \nabla u^0 \cdot \nabla p^0 dx\,.$$

Stoga zaključujemo,

$$J_{sa}^*(\theta) = \int_\Omega j_1(u^0) dx + \int_{\Gamma_N} j_2(u^0) ds - \eta \int_\Omega \theta A^0 \nabla u^0 \cdot \nabla p^0 dx +$$

$$\eta^2 \int_\Omega \frac{1}{2} j_1''(u^0)(u^1)^2 dx + \eta^2 \int_{\Gamma_N} \frac{1}{2} j_2''(u^0)(u^1)^2 ds -$$

$$\eta^2 \int_\Omega \theta A^0 \nabla u^1 \cdot \nabla p^0 dx + \eta^2 \int_\Omega \theta(1-\theta) A^0 M^* A^0 \nabla u^0 \cdot \nabla p^0 dx\,.$$

Lema 3

Funkcija cilja $J^*_{sa}(\theta)$ je Fre\'chet derivabila i njezina derivacija u smjeru $s \in L^\infty(\Omega)$ je dana s

$$\frac{\partial J^*_{sa}}{\partial\theta}(s) = -\eta \int_\Omega sA^0\nabla u^0 \cdot \nabla p^0 dx - \eta^2 \int_\Omega sA^0\nabla u^1 \cdot \nabla p^0 dx +$$

$$\eta^2 \int_\Omega s(1-2\theta)A^0 M^* A^0\nabla u^0 \cdot \nabla p^0 dx - \eta^2 \int_\Omega sA^0\nabla u^0 \cdot \nabla p^1 dx \,,$$

gdje je p^1 drugo pridruženo stanje definirano kao rješenje u $H^1(\Omega)$ zadaće

$$\left\{ \begin{array}{c} -div(A^0\nabla p^1) = j''(u^0)u^1 + div(\theta A^0\nabla p^0) \;\; u \;\Omega \\ p^1 = 0 \;\; na \;\Gamma_D \\ A^0\nabla p_1 \cdot n = j_2''(u^0)u^1 - \theta A^0\nabla p^0 \cdot n \;\; na \;\Gamma_N \end{array} \right. \tag{13}$$

Dokaz: prvo definiramo derivaciju od u^1 po θ u smjeru s, oznaka $z = \frac{\partial u^1}{\partial \theta}(s)$. Lako se vidi da je $u^1 \in H^1(\Omega)$ Fréchet derivabilna po θ i da je

$$\left\{ \begin{array}{c} -\text{div}(A^0 \nabla z) = \text{div}(s A^0 \nabla u^0) \text{ u } \Omega \\ z = 0 \text{ na } \Gamma_D \\ A^0 \nabla z \cdot n = -s A^0 \nabla u^0 \cdot n \text{ na } \Gamma_N \end{array} \right. \tag{14}$$

Tada imamo

$$\frac{\partial J^*_{sa}}{\partial \theta}(s) = -\eta \int_\Omega s A^0 \nabla u^0 \cdot \nabla p^0 dx + \eta^2 \int_\Omega j_1''(u^0) u^1 z dx -$$

$$\eta^2 \int_\Omega s A^0 \nabla u^1 \cdot \nabla p^0 dx + \eta^2 \int_\Omega s(1-2\theta) A^0 M^* A^0 \nabla u^0 \cdot \nabla p^0 dx -$$

$$-\eta^2 \int_\Omega \theta A^0 \nabla z \cdot \nabla p^0 dx + \eta^2 \int_{\Gamma_N} j_2''(u^0) u^1 z dx \, .$$

Da bismo eliminirali z, koristimo pridruženo stanje p^1. Množenjem (13) s z i (14) s p^1 dobijemo

$$\int_\Omega j_1''(u^0)u^1 z dx - \int_\Omega \theta A^0 \nabla p^0 \cdot \nabla z dx + \int_{\Gamma_N} j_2''(u^0)u^1 z ds = -\int_\Omega s A^0 \nabla u^0 \cdot \nabla p^1 dx$$

Funkcija cilja oblika $j(\nabla u)$

$$J(\chi) = \int_\Omega j(\nabla u)dx\,,$$

Koristeći pretpostavku male amplitude, dobijemo

$$\begin{aligned} J_{sa}(\chi) &= \int_\Omega j(\nabla u^0)dx + \eta \int_\Omega j'(\nabla u^0)\cdot\nabla u^1 dx + \\ &\quad \eta^2 \int_\Omega j'(\nabla u^0)\cdot\nabla u^2 + \frac{1}{2} j''(\nabla u^0)\nabla u^1\cdot\nabla u^1 dx \end{aligned}$$

Promatramo optimizacijski problem

$$\inf_{\chi\in\mathcal{U}_{ad}} J_{sa}(\chi) \tag{15}$$

gdje su u^0, u^1 i u^2 rješenja zadaća (4), (5) i (6).

- $\chi_n \overset{*}{\rightharpoonup} \theta(x)$, u $L^\infty(\Omega;[0,1])$
- $u_n^1 \rightharpoonup u^1$ u $H^1(\Omega)$, $u_n^2 \rightharpoonup u^2$ u $H^1(\Omega)$, gdje su u^1 i u^2 rješenja zadaća (8) i (9)

$$\begin{aligned} J_{sa}(\chi_n) &= \int_\Omega j(\nabla u^0)dx + \eta \int_\Omega j'(\nabla u^0)\cdot \nabla u_n^1 dx + \\ & \eta^2 \int_\Omega j'(\nabla u^0)\cdot \nabla u_n^2 + \frac{1}{2} j''(\nabla u^0)\nabla u_n^1 \cdot \nabla u_n^1 dx \end{aligned}$$

Koristeći H-mjere dobijemo:

$$\begin{aligned} \lim_{n\to\infty} J_{sa}(\chi_n) &= J^*_{sa}(\theta,\nu) = \int_\Omega j(\nabla u^0)dx + \eta \int_\Omega j'(\nabla u^0)\cdot \nabla u^1 dx + \\ & \eta^2 \int_\Omega j'(\nabla u^0)\cdot \nabla u^2 + \frac{1}{2} j''(\nabla u^0)\nabla u^1 \cdot \nabla u^1 + \\ & \theta(1-\theta)A^0 N A^0 \nabla u^0 \cdot \nabla u^0 dx\,, \end{aligned}$$

gdje je

$$N = \frac{1}{2}\int_{\mathbb{S}_{N-1}} \frac{j''(\nabla u^0)\xi\cdot\xi}{(A^0\xi\cdot\xi)^2}\xi\otimes\xi\nu(\xi)$$

Propozicija 4

Relaksacija problema (15) je stoga

$$\min_{(\theta,\nu)\in\mathcal{U}^*_{ad}} J^*_{sa}(\theta,\nu)\,. \tag{16}$$

1. *postoji barem jedan minimizator* (θ,ν) *problema (16)*
2. *bilo koji minimizator* (θ,ν) *problema (16) je dobiven pomoću minimizirajućeg niza* χ_n *problema (15) u smislu da* $\chi_n \overset{*}{\rightharpoonup} \theta$ *u* $L^\infty(\Omega)$, $\theta(1-\theta)\nu$ *je H-mjera niza* $(\chi_n - \theta)$ *i* $\lim_{n\to\infty} J_{sa}(\chi_n) = J^*_{sa}(\theta,\nu)$
3. *bilo koji minimizirajući niz* χ_n *problema (15) konvergira u prethodnom smislu k minimizatoru* (θ,ν) *problema (16).*

$$\begin{cases} -\mathrm{div}(A^0\nabla p^0) = -\mathrm{div} j'(\nabla u^0) \text{ u } \Omega \\ p^0 = 0 \text{ na } \Gamma_D \\ A^0\nabla p^0 \cdot n = j'(\nabla u^0) \cdot n \text{ na } \Gamma_N \end{cases}$$

$$\begin{cases} -\mathrm{div}(A^0\nabla p^1) = \mathrm{div}(j''(\nabla u^0)\nabla u^1) - \mathrm{div}(\theta A^0\nabla p^0) \text{ u } \Omega \\ p^1 = 0 \text{ na } \Gamma_D \\ A^0\nabla p^1 \cdot n = (\theta A^0\nabla p^0 - j''(\nabla u^0)\nabla u^1) \cdot n \text{ na } \Gamma_N \end{cases}$$

$$h(\xi) = \frac{(A^0\nabla u^0 \cdot \xi)(A^0\nabla p^0 \cdot \xi)}{A^0\xi \cdot \xi} + \frac{1}{2}\frac{(j''(\nabla u^0)\xi \cdot \xi)(A^0\nabla u^0 \cdot \xi)^2}{(A^0\xi \cdot \xi)^2}$$

$$
\begin{aligned}
J_{sa}^*(\theta) &= \int_\Omega j(\nabla u^0)dx - \eta \int_\Omega \theta A^0 \nabla u^0 \cdot \nabla p^0 dx - \eta^2 \int_\Omega \theta A^0 \nabla u^1 \cdot \nabla p^0 dx + \\
&\quad \frac{\eta}{2} \int_\Omega j''(\nabla u^0) \nabla u^1 \cdot \nabla u^1 dx + \eta^2 \int_\Omega \theta(1-\theta) h(\xi^*) dx
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial J_{sa}^*}{\partial \theta}(s) &= -\eta \int_\Omega s A^0 \nabla u^0 \cdot \nabla p^0 dx + \eta^2 \int_\Omega s A^0 \nabla u^0 \cdot \nabla p^1 dx - \\
&\quad \eta^2 \int_\Omega s A^0 \nabla u^1 \cdot \nabla p^0 dx + \eta^2 \int_\Omega s(1-2\theta) h(\xi^*) dx
\end{aligned}
$$

Napomena 2.

U posebnom slučaju, kada je $j(\nabla u) = |\nabla u|^2$ *i* $A^0 = \alpha \mathrm{I}$, *onda je* $p^0 = \frac{2u^0}{\alpha}$ *i optimalan smjer* $\xi^* \perp \nabla u^0$.

Funkcija cilja oblika $j(A\nabla u)$

$$J(\chi) = \int_\Omega j(A\nabla u)dx\,,$$

Koristeći pretpostavku malih amplituda, dobivamo

$$\begin{aligned} J_{sa}(\chi) \;=\; & \int_\Omega j(A^0\nabla u^0)dx + \eta \int_\Omega j'(A^0\nabla u^0)\cdot(A^0\nabla u^1 + \chi A^0\nabla u^0)dx + \\ & \eta^2 \int_\Omega j'(A^0\nabla u^0)\cdot(A^0\nabla u^2 + \chi A^0\nabla u^1)dx + \\ & \frac{\eta^2}{2}\int_\Omega j''(A^0\nabla u^0)(A^0\nabla u^1 + \chi A^0\nabla u^0)\cdot(A^0\nabla u^1 + \chi A^0\nabla u^0)dx \end{aligned}$$

$$J^*_{sa} \quad (\theta, \nu) = \int_\Omega j(A^0\nabla u^0)dx + \eta \int_\Omega j'(A^0\nabla u^0)\cdot(A^0\nabla u^1 + \theta A^0\nabla u^0)dx -$$
$$\eta^2 \int_\Omega \theta A^0\nabla u^1 \cdot \nabla p^0 dx + \frac{\eta^2}{2}\int_\Omega \theta(j''(A^0\nabla u^0)A^0\nabla u^0)\cdot A^0\nabla u^0 dx +$$
$$\eta^2 \int_\Omega \theta j'(A^0\nabla u^0)\cdot A^0\nabla u^1 dx + \frac{\eta^2}{2}\int_\Omega (j''(A^0\nabla u^0)A^0\nabla u^1)\cdot A^0\nabla u^1 +$$
$$\eta^2 \int_\Omega \theta(j''(A^0\nabla u^0)A^0\nabla u^0)\cdot A^0\nabla u^1 dx +$$
$$\eta^2 \int_\Omega \theta(1-\theta)\int_{\mathbb{S}_{N-1}} h_1(\xi)\nu(dx, d\xi)\,,$$

gdje je

$$h_1(\xi) = \frac{(A^0\nabla u^0\cdot\xi)(A^0\nabla p^0\cdot\xi)}{A^0\xi\cdot\xi} + \frac{1}{2}(j''(A^0\nabla u^0)A^0\xi)\cdot A^0\xi)\left(\frac{A^0\nabla u^0\cdot\xi}{A^0\xi\cdot\xi}\right)^2$$
$$-(j'(A^0\nabla u^0) + j''(A^0\nabla u^0)A^0\nabla u^0)\cdot A^0\xi\frac{A^0\nabla u^0\cdot\xi}{A^0\xi\cdot\xi}$$

$$\left\{\begin{array}{c} -\mathrm{div}(A^0\nabla p^0) = -\mathrm{div}(A^0 j'(A^0\nabla u^0)) \text{ u } \Omega \\ p^0 = 0 \text{ na } \Gamma_D \\ A^0\nabla p^0\cdot n = (A^0 j'(A^0\nabla u^0))\cdot n \text{ na } \Gamma_N \end{array}\right.$$

$$\left\{\begin{array}{c} -\mathrm{div}(A^0\nabla p^1) = \mathrm{div}(\theta A^0(-\nabla p^0 + j'(A^0\nabla u^0) + j''(A^0\nabla u^0)A^0\nabla u^0) + \\ +\mathrm{div}(A^0 j''(A^0\nabla u^0)A^0\nabla u^1) \text{ u } \Omega \\ p^1 = 0 \text{ na } \Gamma_D \\ A^0\nabla p^1\cdot n = \theta A^0(\nabla p^0 - j'(A^0\nabla u^0) - j''(A^0\nabla u^0)A^0\nabla u^0)\cdot n - \\ -(A^0 j''(A^0\nabla u^0)A^0\nabla u^1)\cdot n \text{ na } \Gamma_N \end{array}\right.$$

$$
\begin{aligned}
\frac{\partial J_{sa}^*}{\partial \theta}(s) \quad = \quad & \eta \int_\Omega s j'(A^0 \nabla u^0) \cdot A^0 \nabla u^0 dx - \eta \int_\Omega s A^0 \nabla u^0 \cdot \nabla p^0 dx - \\
& \eta^2 \int_\Omega s A^0 \nabla u^1 \cdot \nabla p^0 dx + \eta^2 \int_\Omega s j'(A^0 \nabla u^0) \cdot A^0 \nabla u^1 dx + \\
& \eta^2 \int_\Omega s (j''(A^0 \nabla u^0) A^0 \nabla u^0) \cdot A^0 \nabla u^1 dx + \\
& \frac{1}{2} \eta^2 \int_\Omega s (j''(A^0 \nabla u^0) A^0 \nabla u^0) \cdot A^0 \nabla u^0 dx + \\
& \eta^2 \int_\Omega s A^0 \nabla u^0 \cdot \nabla p^1 dx + \eta^2 \int_\Omega s (1 - 2\theta) h_1(\xi) dx
\end{aligned}
$$

Napomena 3.

U posebnom slučaju, kada je $j(A\nabla u) = |A\nabla u|^2$ i $A^0 = \alpha\mathrm{I}$, onda je θ konstanta.
Doista, $p^0 = 2\alpha u^0$ i optimalan smjer u svakoj točki je paralelan s ∇u^0.
Zbog toga je $h_1(\xi^*) = -\alpha^2 |\nabla u^0|^2$ i

$$J^*_{sa}(\theta) = \alpha^2 \|\nabla u^0\|^2_{L^2(\Omega)} + \eta^2 \alpha^2 \|\theta \nabla u^0 + \nabla u^1\|^2_{L^2(\Omega)}.$$

Ako je $\theta(x) = \theta_0$, onda je $u^1 = -\theta u^0$ i drugi član u gornjoj formuli je 0.
Time smo pokazali da je $J^*_{sa}(\theta_0)$ zapravo minimum.

Algoritam

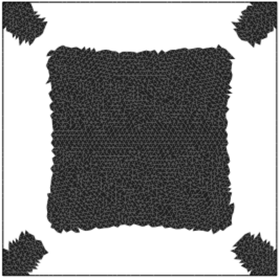
- Inicijalno:
  - izračunati u^0 i p^0
  - izračunati optimalan smjer ξ^*
  - postavi $\theta = \theta_0$ konstanta
  - izračunaj $u^{1,0}$ i $p^{1,0}$
  - evaluiraj $J^*(\theta_0)$.

- Iteracije: za $k \geq 1$ i k manji od maksimalnog broja iteracija
  i) izračunaj gradient $\frac{\partial J^*_{sa}}{\partial \theta}(\theta_{k-1})$ baziran na u^0, p^0, $u^{1,k-1}$ i $p^{1,k-1}$
  ii) s korakom $t_k > 0$ izračunajte

  $$\theta_k = \min(1, \max(0, \tilde{\theta}_k)) \text{ s } \tilde{\theta}_k = \theta_{k-1} - t_k \frac{\partial J^*_{sa}}{\partial \theta}(\theta_{k-1}) + \Lambda_k$$

  gdje je Λ_k Lagrangeov multiplikator za konstantu volumena
  iii) Izračunaj $u^{1,k}$ u evoluiraj $J^*_{sa}(\theta_k)$
  iv) Ako je $J^*_{sa}(\theta_k) < J^*_{sa}(\theta_{k-1})$: izračunaj $p^{1,k}$ i postavi $k = k + 1$. Idi na korak i)
  v) Ako je $J^*_{sa}(\theta_k) \geq J^*_{sa}(\theta_{k-1})$: smanji korak t_k i ponovi korake ii)-v)

Primjer:



$$\min_{(\theta,\nu)\in\mathcal{U}^*_{ad}} \left\{ -\int_\Omega u dx \right\}$$

$$\Omega = (0,1)\times(0,1)$$

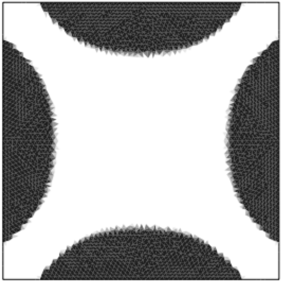
$$\begin{cases} -\text{div}\left((1-0,5\chi)\,\mathrm{I}\nabla u\right) = 1 \text{ u } \Omega \\ u = 0 \text{ na } \partial\Omega \end{cases}$$

$$\mathcal{U}^*_{ad} = \left\{ \begin{array}{c} (\theta,\nu)\in L^\infty(\Omega;[0,1])\times\mathcal{P}(\Omega,\mathbb{S}_{N-1}) \\ \text{t.d. } \int_\Omega \theta(x)dx = 0.5 \end{array} \right\}$$

Primjer:

$$\min_{(\theta,\nu)\in\mathcal{U}_{ad}^*}\left\{\int_\Omega |\nabla u|^2 dx\right\}$$

$$\Omega = (0,1)\times(0,1)$$



$$\begin{cases} -\mathrm{div}\,((1-0,5\chi)\,\mathrm{I}\nabla u) = 1 \text{ u } \Omega \\ u = 0 \text{ na } \partial\Omega \end{cases}$$

$$\mathcal{U}_{ad}^* = \left\{ \begin{array}{c} (\theta,\nu)\in L^\infty(\Omega;[0,1])\times\mathcal{P}(\Omega,\mathbb{S}_{N-1}) \\ \text{t.d. } \int_\Omega \theta(x)dx = 0.4 \end{array} \right\}$$

Hvala na pažnji!