Averaged controllability in a long time horizon

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Outline

- Introduction
- Results
- Examples
- Conclusion

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Averaged control

We consider d realisations of a finite dimensional control systems,

$$\dot{x}_i = A_i x_i + B_i u\,, \quad x_i(0) = x_i^0 \qquad i \in \{1, \cdots, d\}\,. \tag{1}$$

- $A_i \in M_n(\mathbf{R})$, system dynamics
- $B_i \in M_{n,m}(\mathbf{R})$, control operators
- $x_i(t) \in \mathbf{R}^n$, states
- $u(t) \in \mathbf{R}^m$, controls

Definition

The system (1) *is* **controllable in average in some time** $T > 0$ **for the parameters** $\sigma_1, \cdots, \sigma_d$, $\sum \sigma_i = 1$ *if for every* $x_1^0, \cdots, x_d^0 \in \mathbf{R}^n$ *and every* $x^1 \in \mathbf{R}^n$, *there exist a control* $u \in L^2(0,T)^m$ *such that the solution of* (1) *satisfies*

$$\sum_{i=1}^{d} \sigma_i x_i(T) = x^1.$$

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$$\sum_{i=1}^{d} \sigma_i x_i(T) = x^1 = 0.$$

Averaged control

Theorem 1. (Zuazua 14[1])

The system (1) *is controllable in average if and only if the following rank condition is satisfied:*

$$\operatorname{rank}\left[\sum_{i=1}^{d} \sigma_i A_i^k B_i\,, \quad k \in \mathbf{N}\right] = n\,.$$

Control of each component $\not\Rightarrow$ Averaged control $\qquad (A_2 = -A_1)$

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E. Zuazua, Averaged Control. *Automatica* **50(12)** (2014) 3077–3087.

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Averaged control

We introduce the notation

$$\tilde{A} = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_d \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_1 \\ \vdots \\ B_d \end{pmatrix}, \quad \tilde{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{pmatrix},$$

$$L = \begin{pmatrix} \sigma_1 \mathrm{I}_n & \dots & \sigma_d \mathrm{I}_n \end{pmatrix}.$$

The system (1) writes,

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad \tilde{x}(0) = \tilde{x}^0.$$

The average controllability problem is

$$L\tilde{x}(T) = 0.$$

By new notation and Hamilton-Cayley the average control rank condition reduces to

$$\mathrm{rank}\left[L\tilde{A}^k\tilde{B}\,, \quad k \in \{0, \cdots, nd-1\}\right] = n.$$

Long time averaged control

Exact/simultaneous control:

$$\tilde{x}(T) = 0 \Rightarrow \tilde{x}(t) = 0, \quad t > T, \quad \text{with } u = 0.$$

Averaged control:

$$L\tilde{x}(T) = 0 \not\Rightarrow L\tilde{x}(t) = 0, \quad t > T, \quad \text{with} \quad u = 0.$$

(Two systems with different dynamics starting from a same point will produce different trajectories.)

The problem

Can we find a control u such that $L\tilde{x}(t) = 0, \quad t \geq T$?

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Generalisation of averaged control

L - a general linear operator.

In order to achieve long time control we first have to steer the system in time T not just to $\operatorname{Ker} L$, but to its subspace from which it is possible to stay within it (controlled invariant subspace).

TASKS:

1. Identify the supremal controlled invariant subspace $\mathcal{S} \subseteq \operatorname{Ker} L$.
2. Check the conditions under which it is possible to steer the system to $\mathcal{S}$.
3. Construct the optimal long time control.

TASK 1. Construction of the supremal controlled invariant subspace

Consider

$$\dot{x} = Ax + Bu, \qquad x(0) = x^0 \tag{2}$$

and suppose

$$Lx(t) = 0, \qquad t \geq 0.$$

Then

$$LAx(t) = -LBu(t), \qquad t \geq 0,$$

and

$$PLAx(t) = 0, \qquad t \geq 0,$$

P - the orthogonal projector on $\mathrm{Ker}(LB)^\top$.

Thus

$$Lx(t) = 0 \qquad \Longleftrightarrow \qquad \begin{pmatrix} L \\ PLA \end{pmatrix} x(t) = 0$$

TASK 1. Construction of the supremal controlled invariant subspace

Algorithm 1

STEP 0: Set $L_0 = L$.

STEP 1: Set $\Lambda_0 = P_0 L_0 A$, where P_0 is the orthogonal projector on $\mathrm{Ker}(L_0 B)^\top$.

Define $L_1 := \begin{pmatrix} L_0 \\ \Lambda_0 \end{pmatrix}$.

STEP k+1: Set $\Lambda_k = P_k L_k A$, where P_k is the orthogonal projector on $\mathrm{Ker}(L_k B)^\top$.

Define $L_{k+1} := \begin{pmatrix} L_0 \\ \Lambda_k \end{pmatrix}$.

The algorithm stops when $\mathrm{Ker}\, L_{K+1} = \mathrm{Ker}\, L_K$.

Theorem 2.

$\mathrm{Ker}\, L_K$ *i the supremal invariant subspace of* $\mathrm{Ker}\, L$.

TASK 2

Under which conditions we can reach $\operatorname{Ker} L_K$?

When it is possible to steer the system to $\operatorname{Ker}\Lambda$, Λ – arbitrary?

Theorem 3. (cf. Kreindler, Sarachik 64)[2]

For every $x^0 \in \mathbf{R}^n$ and $\bar{x}^1 \in \operatorname{Ran}\Lambda$ there exists a control $u \in L^2(0,T)^m$ steering the solution x of

$$\dot{x} = Ax + Bu, \qquad x(0) = x^0$$

to some $x(T) \in \mathbf{R}^n$ such that $\Lambda x(T) = \bar{x}^1$, if and only if A, B and Λ satisfy:

$$\operatorname{rank}\Lambda\left(A^0 B \ \ldots \ A^{n-1}B\right) = \operatorname{rank}\Lambda.$$

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Kreindler, E. and Sarachik, P. E., On the concepts of controllability and observability of linear systems, *IEEE Trans. Automatic Control* **AC-9)** (1964) 129–136.

Long time control result

Theorem 4.

For every $x^0 \in \mathbf{R}^n$ there exists a control $u \in L^\infty(\mathbf{R}_+)^m$ such that the solution to the system

$$\dot{x} = Ax + Bu, \qquad x(0) = x^0$$

satisfies $Lx(t) = 0, t \geq T$ if and only if

$$\operatorname{rank} L_K \begin{pmatrix} A^0 B & \dots & A^{n-1} B \end{pmatrix} = \operatorname{rank} L_K \,,$$

where L_K is the operator constructed by the Algorithm 1.

TASK 3. Norm optimal controls

THE AIM

– given two positive times T_0 and T_1,
– given initial condition $x^0 \in \mathbf{R}^n$,
– given an operator $L \in M_{q,n}(\mathbf{R})$
– find the control of minimal L^2 norm such that the solution x of (2) satisfies

$$Lx(t) = 0, \qquad t \in [T_0, T_0 + T_1]\,.$$

$J_0(x^1)$ — the minimal control norm steering x^0 to x^1 in time T_0

$J_1(x^1)$ — the minimal control norm keeping the solution within $\mathrm{Ker}\, L$ for $t \in [T_0, T_0 + T_1]$.

We have to minimise $J_0 + J_1$ over $x^1 \in \mathrm{Ker}\, L_K \cap R_{T_0}(x^0)$.

$R_{T_0}(x^0)$ – the set of reachable points from x^0 in time $T_0 > 0$.

$J_0(x^1)$ – minimal norm steering the system to x^1

The minimal norm control steering the system to x^1 is

$$u(t) = B^\top e^{(T_0 - t)A^\top} Q_{T_0} \left(x^1 - e^{T_0 A} x^0 \right) ,$$

where Q_{T_0} is the *inverse* of the Gramian Γ_{T_0}:

$$Q_{T_0} \Gamma_{T_0} p^1 = p^1, \quad p^1 \in \mathrm{Ran}\, \Gamma_{T_0} .$$

(We do not assume controllability of (A, B)).
Minimal norm is

$$J_0(x^1; T_0) := \|u\|^2_{L^2(0,T_0)} = \left(x^1 - e^{T_0 A} x^0 \right)^\top Q_{T_0} \left(x^1 - e^{T_0 A} x^0 \right) .$$

$J_1(x^1)$ – minimal norm keeping the solution within $\operatorname{Ker} L$

Decompose

$$u(t) = u_0(t) + (LB)^\top v_0(t), \quad u_0(t) \in \operatorname{Ker} LB.$$

$Lx = const$ implies

$$-LAx = (LB)(LB)^\top v_0.$$

Consequently,

$$v_0(x) = -MLAx,$$

where M is the *inverse* of $(LB)(LB)^\top$:

$$M(LB)(LB)^\top w = w, \quad w \in \operatorname{Ran} LB.$$

It remains to detect u_0.

$J_1(x^1)$ – minimal norm keeping the solution within $\mathrm{Ker}\, L$

We show

$$u_0 = KB^\top E(t),$$

where $E \in M_n(\mathbf{R})$ is solution of the backward Riccati equation:

$$\dot{E} = (LA)^\top MLA - \left(A - B(LB)^\top MLA\right)^\top E - E\left(A - B(LB)^\top MLA\right) \\ - EBB^\top E, \\ E(T) = 0.$$

and $K : \mathbf{R}^{d(LB)} \to \mathbf{R}^m$ is an isometry operator such that $\mathrm{Ran}\, K = \mathrm{Ker}\, LB$. Thus, optimal control is a feedback control of the form

$$u(t) = \left(KB^\top E(t) - (LB)^\top MLA\right)x(t).$$

Furthermore

$$J_1(x^0; T) := -\frac{1}{2}(x^0)^\top E(0)x^0$$

and $E(0)$ is a non-positive matrix.

Optimal long time control

Denote

$$\mathcal{U} = \left\{ u \in L^2(0, T_0 + T_1)^m \ \mid \ Lx(t) = 0, \forall t \in [T_0, T_0 + T_1] \right\}.$$

According to Bellman principle

$$\min_{u \in \mathcal{U}} \frac{1}{2} \int_0^{T_0+T_1} |u(t)|^2 \, dt = \min_{x^1 \in \mathrm{Ker}\, L_K \cap R_{T_0}(x^0)} \left(J_0(x^1; T_0) + J_1(x^1, T_1) \right),$$

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Long time averaged control

We take

$$L = \begin{pmatrix} \mathrm{I}_n & \mathrm{I}_n \end{pmatrix}$$

and the system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad \tilde{x}(0) = \tilde{x}^0 \tag{3}$$

for

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \tilde{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

The long time averaged control problem is

$$L\tilde{x}(Tt) = 0, \quad t > T.$$

The operator constricted by Algorithm 1 is

$$L_n = \begin{pmatrix} I_n & I_n \\ P_B D & -P_B D \\ P_B DS & -P_B DS \\ \vdots & \vdots \\ P_B DS^{n-1} & -P_B DS^{n-1} \end{pmatrix},$$

with

- $D = \frac{1}{2}(A_1 - A_2)$, $S = \frac{1}{2}(A_1 + A_2)$
- P_B the projector on $\mathrm{Ker}(LB)^\top = \mathrm{Ker}\, B^\top$.

Comparison of different control notions

Denote

$$K = \left(\tilde{B},\ \tilde{A}\tilde{B},\ \cdots,\ \tilde{A}^{2n-1}\tilde{B}\right).$$

The system (3) is simultaneously controllable if and only if

$$r(K) = 2n.$$

The system (3) has the long time averaged property if and only if

$$r(L_n K) = r(L_n),$$

with L_n constructed by the Algorithm 1.

The system (3) is averaged controllable if and only if

$$r(K) = n.$$

Simultaneous $\Longrightarrow$ Long time averaged $\Longrightarrow$ Averaged

Example 1

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We have

$$K = \left(\tilde{B},\ \tilde{A}\tilde{B},\ \tilde{A}^2\tilde{B},\ \tilde{A}^3\tilde{B}\right) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

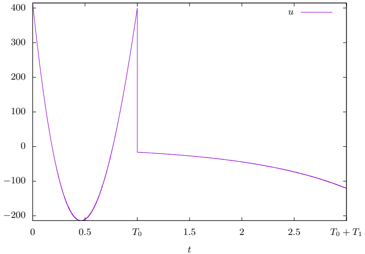
$$\operatorname{rank} K = 3 < 4 \quad \Longrightarrow \quad \text{No simultaneous controllability.}$$

$$\operatorname{rank} L_2 K = 3 = \operatorname{rank} L_2 \quad \Longrightarrow \quad \text{Long time averaged controllable.}$$

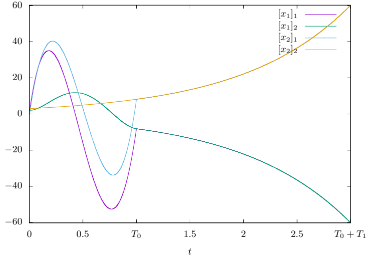
$\operatorname{Ker} L_2 = \left(1,\ 1,\ -1,\ -1\right)^\top \mathbf{R}$.

Take $\tilde{x}^0 = (1,\ 2,\ -1,\ 3)^\top$.

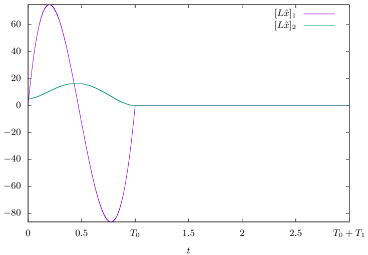
Example 1 - cont.



(a) Control



(b) State



(c) Average of the state

Example 2

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We have

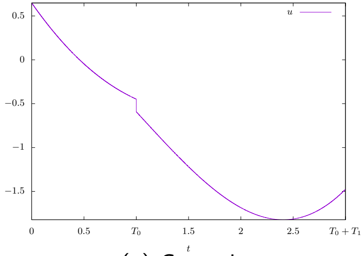
$$K = \left(\tilde{B},\ \tilde{A}\tilde{B},\ \tilde{A}^2\tilde{B},\ \tilde{A}^3\tilde{B}\right) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$\operatorname{rank} K = 4 \quad \Longrightarrow \quad$ The system is simultaneous controllable.

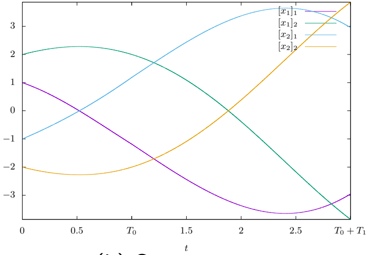
$d(L_2) = 2$

Take $\tilde{x}^0 = (1,\ 2,\ -1,\ -2)^\top$.

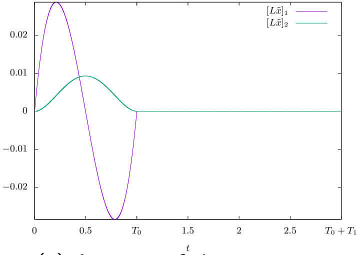
Example 2 - cont.



(a) Control



(b) State



(c) Average of the state

Outline

Conclusion

Long time averaged control

- A new control notion for parameter dependent problems,
- Between simultaneous and averaged control,
- Generalisation to arbitrary operators.

Open questions

- The results provided for a finite number of parameters.
  What for an infinite number of parameters (either discrete or continuous) ?
  In this case, we consider a system

$$\dot{x}_\zeta(t) = A_\zeta x_\zeta(t) + B_\zeta u(t), \quad x_\zeta(0) = x_\zeta^0,$$

  $\zeta \in \Omega$ – an unknown parameter and with $(\Omega, \mathcal{F}, \mu)$ a probability space.
  Assuming $\int_\Omega x_\zeta^0 \, d\mu_\zeta = 0$ under which conditions does it exist a control u independent of ζ such that $\int_\Omega x_\zeta(t) \, d\mu_\zeta = 0$ for every $t > 0$?
- A similar question can be addressed for PDEs.
  In this case, the algebraic relation we used surely fails.

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Thanks for your attention!