

Stabilizing Transmissions and Delays in Nonlinear Networked Control Systems: Hybrid Systems with Memory and Lyapunov Approach

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Outline

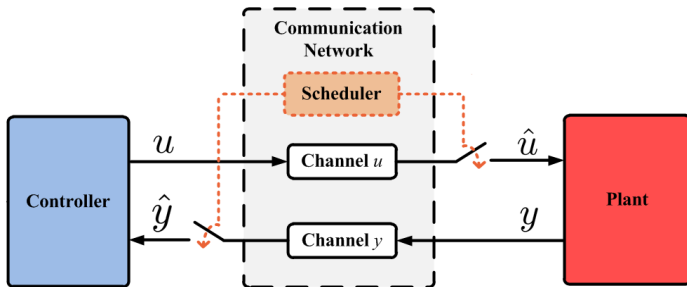
- 1 Introduction
 - Motivation
 - Preliminaries
 - Contributions
- 2 Problem Formulation
 - Rewriting Dynamics
- 3 Methodology
 - Assumptions
 - Main Result
- 4 Example
- 5 Future Work

From Analog to Digital and Networked World

- many processes exhibit both **continuous** and **discrete** dynamics – **hybrid dynamics**¹
- colliding rigid bodies and electrical circuits with active components
- **digital technology** for monitoring and controlling purposes
- communication/measuring devices, sampling units and logic devices
- **delays** in control processes per se and owing to digital technology

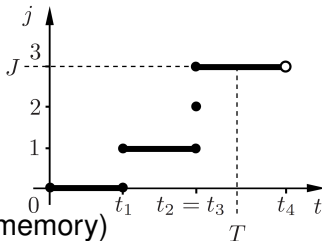
¹Rafal Goebel, Ricardo G. Sanfelice, and Andrew R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

Real-Life Control System

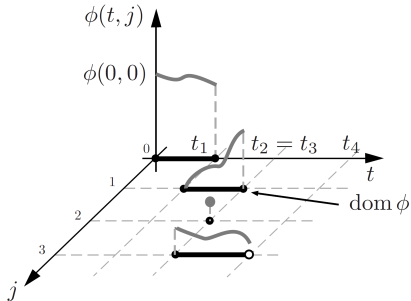


Hybrid Systems with Memory

- hybrid time domain (with memory)



- hybrid arc (with memory)



Hybrid Systems with Memory

- given $d \geq 0$, we use \mathcal{M}^d to denote the **collection of hybrid memory arcs** ϕ satisfying $s \geq -d$ for all $(s, k) \in \text{dom } \phi$
- **system state** $x_t : \text{dom}_{\geq 0}(x) \rightarrow \mathcal{M}^d$ defined by

$$x_t(s, k(s)) := x(t + s, j + k(s))$$

for all $(s, k(s)) \in \text{dom } x_t$, where

$$k(s) := \max\{k : (s, k) \in \text{dom } x_t\} \text{ and } \text{dom } x_t := \{(s, k) \in \mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0} : (t + s, j + k) \in \text{dom } x, s \in [-d, 0]\}$$

- d is the **maximum value** of all delay phenomena in the dynamics

Hybrid Systems with Memory

Definition

A **hybrid system with memory** d is defined by a 4-tuple $\mathcal{H}_{\mathcal{M}}^d = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ as follows:

- a set $\mathcal{C} \subset \mathcal{M}^d$, called the **flow set**,
- a function $\mathcal{F} : \mathcal{M}^d \rightarrow \mathbb{R}^{n_x}$, called the **flow map**,
- a set $\mathcal{D} \subset \mathcal{M}^d$, called the **jump set**, and
- a function $\mathcal{G} : \mathcal{M}^d \rightarrow \mathbb{R}^{n_x}$, called the **jump map**.

²Jun Liu and Andrew R. Teel. “Lyapunov-Based Sufficient Conditions for Stability of Hybrid Systems With Memory”. In: *IEEE Trans. on Automatic Control* 61.4 (2016), pp. 1057–1062.

Hybrid Systems with Memory

Definition

A hybrid arc is a **solution** to the hybrid system $\mathcal{H}_{\mathcal{M}}^d$ if the **initial data** $x_0 \in \mathcal{C} \cup \mathcal{D}$ and

- (i) for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in \text{dom}_{\geq 0}(x)$

$$x_t \in \mathcal{C}, \quad \dot{x}(t, j) = \mathcal{F}(x_t),$$

- (ii) for all $(t, j) \in \text{dom}_{\geq 0}(x)$ such that $(t, j + 1) \in \text{dom}_{\geq 0}(x)$,

$$x_t \in \mathcal{D}, \quad x(t, j + 1) = \mathcal{G}(x_t).$$

Set pre-UGAS

Definition

Let $\mathcal{W} \in \mathbb{R}^{n_x}$ be a **closed set**. The set \mathcal{W} is **Uniformly Globally pre-Asymptotically Stable** (UGpAS) for system $\mathcal{H}_{\mathcal{M}}^d$ if there exists a \mathcal{KL} function β such that any solution x to $\mathcal{H}_{\mathcal{M}}^d$ satisfies

$$\|x(t, j)\|_{\mathcal{W}} \leq \beta\left(\|x_0\|_{\mathcal{W}}, t + j\right), \quad \forall (t, j) \in \text{dom}_{\geq 0}(x),$$

where $\|z\|_{\mathcal{W}} := \inf_{y \in \mathcal{W}} \|y - z\|$ for $z \in \mathbb{R}^{n_x}$ and
 $\|\phi\|_{\mathcal{W}} := \sup_{\substack{(s, k) \in \text{dom } \phi \\ s \in [-d, 0]}} \|\phi(s, k)\|_{\mathcal{W}}$ for $\phi \in \mathcal{M}^d$.

Contributions

- 1 hybrid systems-with-memory modeling for a large class of NCSs

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Contributions

- 1 hybrid systems-with-memory modeling for a large class of NCSs
- 2 more general and less conservative sufficient conditions for finding stabilizing transmission intervals and delays in nonlinear NCSs
- 3 the first Lyapunov-based small-gain theorem for hybrid systems with memory
- 4 novel robustness analyses for LTI control systems involving plant input delays, output feedback, observer-predictor-based dynamic controllers and network-induced distributed delays

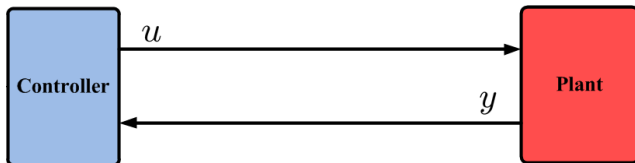
Stabilizing Transmission Intervals and Delays

- nonlinear delayed **plant**

$$\begin{aligned}\dot{x}_p &= f_p(t, x_{p_t}, u_t), \\ y &= g_p(t, x_{p_t}),\end{aligned}\tag{1}$$

- nonlinear delayed **controller**

$$\begin{aligned}\dot{x}_c &= f_c(t, x_{c_t}, y_t), \\ u &= g_c(t, x_{c_t}),\end{aligned}\tag{2}$$



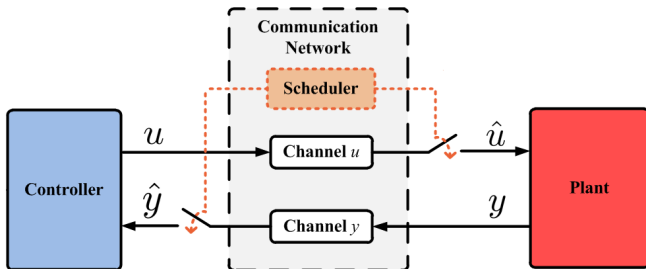
Introduce Communication Network

- ℓ communication channels with **discrete delays**

$$(y(t - d_i(t)), u(t - d_i(t)))_i \quad (3)$$

or **distributed delays**

$$\int_{t-d_i(t)}^t \Gamma_i(t, s, (y(s), u(s)))_i ds \quad (4)$$



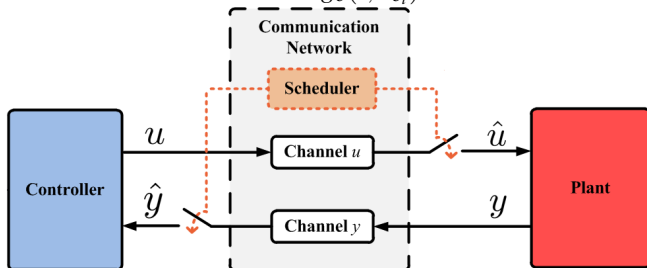
Introduce Communication Network

- nonlinear delayed plant

$$\begin{aligned}\dot{x}_p &= f_p(t, x_{p_t}, \hat{u}_t), \\ y &= g_p(t, x_{p_t}),\end{aligned}\quad (5)$$

- nonlinear delayed controller

$$\begin{aligned}\dot{x}_c &= f_c(t, x_{c_t}, \hat{y}_t), \\ u &= g_c(t, x_{c_t}).\end{aligned}\quad (6)$$

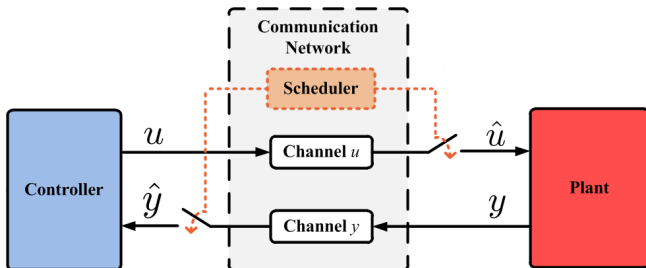


Stabilizing Transmission Intervals and Delays

- define the **error vector**

$$e(t) = \begin{bmatrix} e_y(t) \\ e_u(t) \end{bmatrix} := \begin{bmatrix} \hat{y}(t) - y^*(t) \\ \hat{u}(t) - u^*(t) \end{bmatrix}, \quad (7)$$

where the components of $(y^*(t), u^*(t))$ are given by (3)-(4)



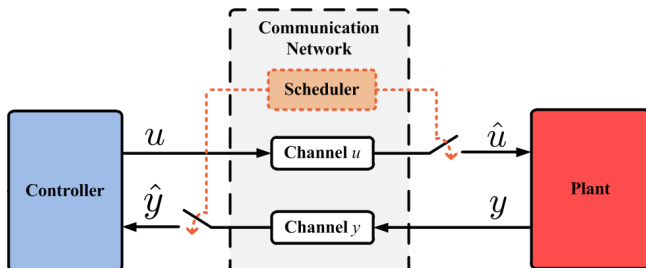
Stabilizing Transmission Intervals and Delays

- \hat{y} and \hat{u} are updated at instants t_κ , $\kappa \in \mathbb{Z}_{\geq 0}$, such that $0 < \epsilon \leq t_{\kappa+1} - t_\kappa \leq \tau_{\text{MATI}}$, yielding

$$\begin{aligned}\hat{y}(t_\kappa^+) &= y^*(t_\kappa) + h_y(\kappa, e(t_\kappa)), \\ \hat{u}(t_\kappa^+) &= u^*(t_\kappa) + h_u(\kappa, e(t_\kappa)),\end{aligned}\tag{8}$$

- consequently

$$e_i(t_\kappa^+) = \mathbf{0}_{n_{e_i}}\tag{9}$$



Hybrid Systems with Memory

- altogether³:

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{e} \\ \dot{\tau}_1 \\ \dot{\kappa} \\ \dot{\tau}_2 \end{bmatrix} = \begin{bmatrix} f(x_t, e_t, \tau_{2t}) \\ g(x_t, e_t, \tau_{2t}) \\ 1 \\ 0 \\ 1 \end{bmatrix} = \mathcal{F}(\xi_t), \underbrace{\tau_{1t}(0, 0) \in [0, \tau_{\text{MATI}}]}_c, \quad (10)$$

$$\xi^+ = \begin{bmatrix} x^+ \\ e^+ \\ \tau_1^+ \\ \kappa^+ \\ \tau_2^+ \end{bmatrix} = \begin{bmatrix} x_t(0, 0) \\ h(\kappa_t(0, 0), e_t(0, 0)) \\ 0 \\ \kappa_t(0, 0) + 1 \\ \tau_{2t}(0, 0) \end{bmatrix} = \mathcal{G}(\xi_t), \underbrace{\tau_{1t}(0, 0) \in [\epsilon, \tau_{\text{MATI}}]}_D, \quad (11)$$

where $\xi := (x, e, \tau_1, \kappa, \tau_2)$

³Contribution 1

Hybrid Systems with Memory

$$f(x_t, e_t, \tau_{2_t}) \stackrel{(5),(6)}{:=} \begin{bmatrix} f_p(\tau_{2_t}, x_{p_t}, \underbrace{g_{c_t}(\tau_{2_t}, x_{c_t}) + e_{u_t}}_{=\hat{u}_t \text{ using (6) and (7)}}) \\ f_c(\tau_{2_t}, x_{c_t}, \underbrace{g_{p_t}(\tau_{2_t}, x_{p_t}) + e_{y_t}}_{=\hat{y}_t \text{ using (5) and (7)}}) \end{bmatrix} \quad (12)$$

$$h(\kappa_t(0, 0), e_t(0, 0)) := \begin{bmatrix} h_y(\kappa_t(0, 0), e_t(0, 0)) \\ h_u(\kappa_t(0, 0), e_t(0, 0)) \end{bmatrix} \quad (13)$$

$$g(x_t, e_t, \tau_{2_t}) \stackrel{(7)}{:=} \begin{bmatrix} \underbrace{\hat{f}_p(\tau_{2_t}, x_{p_t}, x_{c_t}, g_{p_t}(x_{p_t}) + e_{y_t}, g_{c_t}(x_{c_t}) + e_{u_t})}_{\text{model-based estimator}} - \dot{y}^*(\tau_{2_t}(0, 0)) \\ \hat{f}_c(\tau_{2_t}, x_{p_t}, x_{c_t}, g_{p_t}(x_{p_t}) + e_{y_t}, g_{c_t}(x_{c_t}) + e_{u_t}) - \dot{u}^*(\tau_{2_t}(0, 0)) \end{bmatrix} \quad (14)$$

Stabilizing Transmission Intervals and Delays

Problem

Find τ_{MATI} that renders the **set** $\{(x, e, \tau_1, \kappa, \tau_2) : x = \mathbf{0}_{n_x}, e = \mathbf{0}_{n_e}\}$ **UGpAS** for the **closed-loop** hybrid dynamics with memory (10)-(11).

Assumptions

Definition

The **protocol** given by $h := (h_y, h_u)$ is **UGES** if there exist a function $W : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, real number $\rho \in [0, 1)$ and functions $\underline{\alpha}$, $\bar{\alpha} \in \mathcal{K}_\infty$ such that the following holds for all $\kappa \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{R}^{n_e}$:

$$\underline{\alpha}(\|e\|) \leq W(\kappa, e) \leq \bar{\alpha}(\|e\|), \quad (15)$$

$$W(\kappa + 1, h(\kappa, e)) \leq \rho W(\kappa, e), \quad (16)$$

Common UGES protocols are Round Robin (RR) and Try Once Discard (TOD).

Assumptions

Assumption

There exist a locally Lipschitz **Lyapunov-Krasovskii functional** $V : \mathcal{M}_{n_x}^d \rightarrow \mathbb{R}_{\geq 0}$, a continuous functional $H : \mathcal{M}_{n_x}^d \rightarrow \mathbb{R}_{\geq 0}$, real numbers $L_i, J_j \geq 0$, and a continuous positive-definite function $\varrho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that:

- (i) for all $e \in \mathbb{R}^{n_e}$, $\kappa \in \mathbb{Z}_{\geq 0}$, and $x_t \in \mathcal{M}_{n_x}^d$ the inequality

$$\dot{V}(x_t, \dot{x}_t) \leq -\varrho(\|x_t(0, 0)\|) - H^2(x_t) + \gamma^2 W^2(\kappa, e) \quad (17)$$

holds almost everywhere along the solutions of the **nominal system** \mathcal{H}_n , and

- (ii) for all nonnegative hybrid memory arcs $\kappa_t \in \mathcal{M}_{n_\kappa}^d$, all $x_t \in \mathcal{M}_{n_x}^d$ and $e_t \in \mathcal{M}_{n_e}^d$ the inequality (18) holds almost everywhere along the solutions of the **error system** \mathcal{H}_e , where $\Delta_i(t)$'s and $\delta_j(t)$'s are differentiable satisfying $\Delta_i(t) < \bar{\Delta}_i$, $|\dot{\Delta}_i(t)| \leq \check{\Delta}_i < 1$, $\delta_j(t) < \bar{\delta}_j$ and $|\dot{\delta}_j(t)| \leq \check{\delta}_j < 1$.

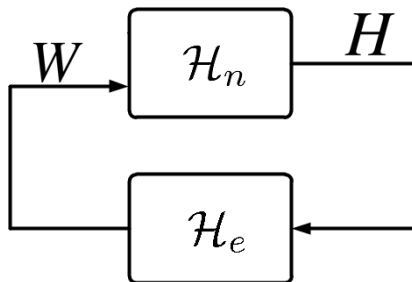
Assumptions

- the error dynamics upper bound⁴:

$$\begin{aligned}
 & \left\langle \frac{\partial W(\kappa_t(0, 0), e_t(0, 0))}{\partial e}, g(x_t, e_t, \tau_{2t}) \right\rangle \leq L_0 W(\kappa_t(0, 0), e_t(0, 0)) + \\
 & + \sum_{i=1}^n L_i W(\kappa_t(-\Delta_i(t), k(-\Delta_i(t))), e_t(-\Delta_i(t), k(-\Delta_i(t)))) + \\
 & + \sum_{j=1}^m J_j \int_{t-\delta_j(t)}^t W(\kappa_t(s, k(s)), e_t(s, k(s))) ds + H(x_t) \quad (18)
 \end{aligned}$$

⁴Contribution 

Assumptions



Main Result


Theorem

Suppose Assumption 1 is satisfied. In addition, suppose that there exist regular enough **function** $\psi : [0, \tau_{\text{MATI}}] \rightarrow \mathbb{R}_{>0}$ and constants $p_i, \mu_i, r_i, \nu_j \geq 0$ such that

(iii) the inequality

$$\rho^2 \leq \frac{\psi(\tau_{\text{MATI}})}{\psi(0)}, \text{ and} \quad (19)$$

(iv) the matrix inequality (20) with expressions (21)-(23) hold, then the set $\mathcal{W} := \{(x, e, \tau_1, \kappa, \tau_2) : x = \mathbf{0}_{n_x}, e = \mathbf{0}_{n_e}\}$ is **UGPAS** for the **closed-loop** system (10)-(11).⁵

⁵Contribution 

Main Result

$$\begin{bmatrix}
 \psi_0 & \psi L_1 + L_0 L_1 \Xi + r_1 & 0 & \cdots & \psi L_n + L_0 L_n \Xi + r_n & 0 & \psi J_1 + L_0 J_1 \Xi & \cdots & \psi J_m + L_0 J_m \Xi & \psi + L_0 \Xi \\
 \psi L_1 + L_0 L_1 \Xi + r_1 & \psi_1 & r_1 & \cdots & L_1 L_n \Xi & 0 & L_1 J_1 \Xi & \cdots & L_1 J_m \Xi & L_1 \Xi \\
 0 & r_1 & \frac{r_1}{\psi_1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 \psi L_n + L_0 L_n \Xi + r_n & L_1 L_n \Xi & 0 & \cdots & \psi_n & r_n & L_n J_1 & \cdots & L_n J_m & L_n \Xi \\
 0 & 0 & 0 & \cdots & r_n & \frac{r_n}{\psi_n} & 0 & \cdots & 0 & 0 \\
 \psi J_1 + L_0 J_1 \Xi & L_1 J_1 \Xi & 0 & \cdots & L_n J_1 & 0 & \Phi_1 & \cdots & J_1 J_m & J_1 \Xi \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 \psi J_m + L_0 J_m \Xi & L_1 J_m \Xi & 0 & \cdots & L_n J_m & 0 & J_1 J_m & \cdots & \Phi_m & J_m \Xi \\
 \psi + L_0 \Xi & L_1 \Xi & 0 & \cdots & L_n \Xi & 0 & J_1 \Xi & \cdots & J_m \Xi & -1 + \Xi
 \end{bmatrix} < 0 \quad (20)$$

$$\psi_0 = \gamma^2 + 2\psi L_0 + \dot{\psi} + \sum_{i=1}^n (p_i + \mu_i + r_i \bar{\Delta}_i^2 - r_i) + \sum_{j=1}^m \nu_j \bar{\delta}_j^2, \quad (21)$$

$$\psi_i = -2r_i + L_i^2 \Xi - \mu_i (1 - \check{\Delta}_i) \quad (22)$$

$$\bar{\psi}_i = -p_i - r_i, \quad \Phi_j = -\nu_j (1 - \check{\delta}_j) + J_j^2 \Xi, \quad \Xi = \sum_{i=1}^n \bar{\Delta}_i^2 r_i \quad (23)$$

Main Result

Corollary

Suppose that there exist constants $\mu_i, \nu_j \geq 0$ such that

(iii') the inequality

$$\rho^2 \leq M,$$

where $M := e^{-\sigma \tau_{MATI}}$ and $\sigma \geq 0$, and

(iv') the matrix inequality

$$\begin{bmatrix} N & L_1 M & \dots & L_n M & J_1 M & \dots & J_m M \\ L_1 M & -\mu_1(1-\check{\Delta}_1) & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 & 0 & 0 \\ L_n M & 0 & 0 & -\mu_n(1-\check{\Delta}_n) & 0 & 0 & 0 \\ J_1 M & 0 & 0 & 0 & -\nu_1(1-\check{\delta}_1) & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & \ddots & 0 \\ J_m M & 0 & 0 & 0 & 0 & 0 & -\nu_m(1-\check{\delta}_m) \end{bmatrix} < 0,$$

where $N := \gamma^2 + (2L_0 - \sigma)M + M^2 + \sum_{i=1}^n \mu_i + \sum_{j=1}^m \nu_j \check{\delta}_j^2$,

hold, then the set $\mathcal{W} := \{(x, e, \tau_1, \kappa, \tau_2) : x = \mathbf{0}_{n_x}, e = \mathbf{0}_{n_e}\}$ is UGpAS for (10)-(11).

Example

- plant with delayed input:

$$\dot{x}_p(t) = A_p x_p(t) + B_p u(t - \delta), \quad (24a)$$

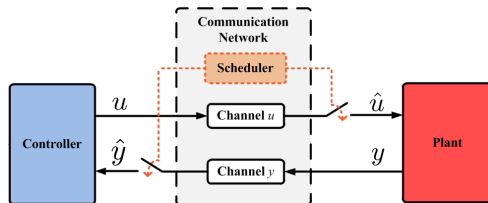
$$y(t) = C_p x_p(t), \quad (24b)$$

- dynamic output feedback observer-predictor controller:

$$\dot{x}_{c1}(t) = A_c x_{c1}(t) + B_c \left[e^{A_p \delta} x_{c2}(t) + \int_{t-\delta}^t e^{A_p(t-\theta)} B_p u(\theta) d\theta \right], \quad (25a)$$

$$\dot{x}_{c2}(t) = A_p x_{c2}(t) + B_p u(t - \delta) + L [y(t) - C_p x_{c2}(t)], \quad (25b)$$

$$u(t) = C_c x_{c1}(t), \quad (25c)$$

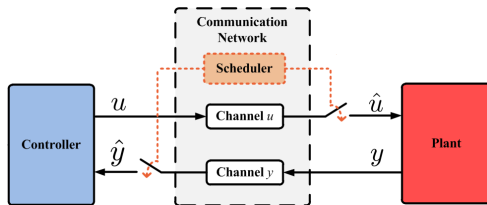


Communication Network

- error vector:

$$e(t) = \begin{bmatrix} e_y(t) \\ e_u(t) \end{bmatrix} = \begin{bmatrix} \hat{y}(t) - \int_{t-\delta}^t y(s) ds \\ \hat{u}(t) - u(t) \end{bmatrix}$$

and $\ell = 2$

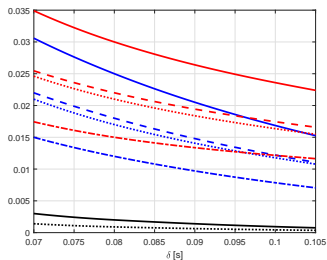
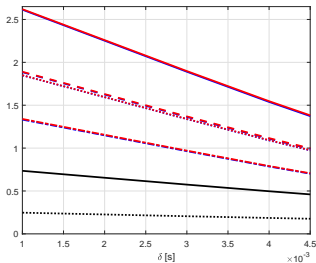
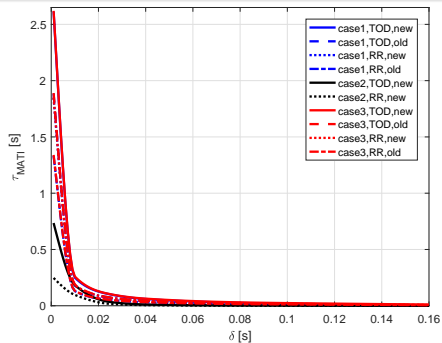


Three Cases

- **Case 1:** u and y in (24a) and (25b), respectively, are replaced with \hat{u} and \hat{y} , respectively,
- **Case 2:** Case 1 together with \hat{u} instead of u in (25a)-(25b), and
- **Case 3:** Case 2 augmented with the model-based predictor $\dot{\hat{u}} = C_c B_c \int_{t-\delta}^t e^{A_p(t-\theta)} B_p \hat{u}(\theta) d\theta$.⁶

⁶Contribution  & 

Numerical Results



Future Work

- more generic Lypunov-Krasovskii functionals
- optimal parameters in the presented stability conditions
- event- and self-triggering

Thanks

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**Questions? Comments?
Suggestions?**

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Thank You for Your attention!!