Greedy optimal control for elliptic equations

Applications to turnpike control

Martin Lazar

University of Dubrovnik

in collaboration with:

Víctor Hernández-Santamaría (*DeustoTech, University of Deusto*)

Enrique Zuazua *(University of Deusto, UAM)*

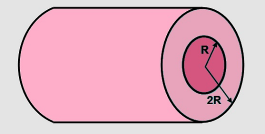
April, 2018

hrzz Hrvatska zaklada za znanost

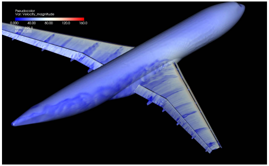
ConDyS

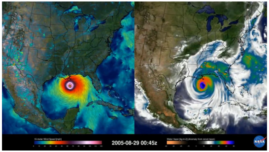
Parameter dependent problems

Real life applications (may) depend on a large number of parameters



examples: thickness, conductivity, density, length, humidity, pressure, curvature,. . .





Parameter dependent problems (Cont.)

- When dealing with applications and simulations, we would like to explore within different parameter configurations.
- From the control point of view, this implies solving a different problem for each configuration.
- Computationally expensive.

Our goal

Apply greedy theory to have a robust and fast numerical solvers.

Parameter dependent control problem

$\Omega \subset \mathbb{R}^N, \quad \omega \subset \Omega.$

Consider the system

$$\begin{cases} -\text{div}(a(x,\nu)\nabla y) + c\,y = \chi_\omega u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

◦ ν is a parameter ◦ $u \in L^2(\omega)$ is a control ◦ $c = c(x) \in L^\infty(\Omega)$

Optimal control problem (OCP$_\nu$)

$$\min_{u \in L^2(\omega)} J_\nu(u) = \frac{1}{2}|u|^2_{L^2(\omega)} + \frac{\beta}{2}\|y - y_d\|^2_{L^2(\Omega)},$$

Parameter dependent control problem (cont.)

Optimal control problem (OCP$_\nu$)

$$\min_{u \in L^2(\omega)} J_\nu(u) = \frac{1}{2}|u|^2_{L^2(\omega)} + \frac{\beta}{2}\|y - y_d\|^2_{L^2(\Omega)},$$

$\exists!$ optimal solution is well-known (Lions, Tröltzsch,...)

Characterization: optimal pair $(\bar{u}, \bar{y})$

$$\bar{u} = -\chi_\omega \bar{q}$$

$$\begin{cases} -\mathrm{div}(a(x,\nu)\nabla \bar{y}) + c\,\bar{y} = -\chi_\omega \bar{q}, & \text{in } \Omega, \\ -\mathrm{div}(a(x,\nu)\nabla \bar{q}) + c\,\bar{q} = \beta\,(\bar{y} - y^d), & \text{in } \Omega, \\ \bar{y} = \bar{q} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

As the state y depends on ν, also the control u depends on ν.

Parameter dependent control problem (cont.)

$$\begin{cases} -\mathrm{div}(a(x,\nu)\nabla y) + c\,y = \chi_\omega u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases}$$

From the practical point of view,

- Measure parameter ν and determine u_ν

$$\min_{u \in L^2(\omega)} J_\nu(u) = \frac{1}{2}|u|^2_{L^2(\omega)} + \frac{\beta}{2}\|y - y_d\|^2_{L^2(\Omega)},$$

  using classical methods (iterative methods, ...)

- Repeat the process for each new value of ν.

Can we do it better?

Greedy control

Assume that ν ranges within a compact set $\mathcal{K} \subset \mathbb{R}^d$ and $a_\nu = a(x, \nu)$ are bounded functions satysfing

$$0 < a_1 \leq a_\nu \leq a_2, \qquad \nu \in K.$$

In this way, we ensure that each control can be uniquely determined by

$$\bar{u}_\nu = -\chi_\omega \bar{q}$$

where $(\bar{y}, \bar{q})$ solve the optimality system (8). Consider the set of controls $\bar{u}_\nu$ for each possible value $\nu \in \mathcal{K}$. That is,

$$\bar{\mathcal{U}} = \{\bar{u}_\nu : \nu \in \mathcal{K}\}$$

The idea

To determine a finite number of values of ν that yield the best possible approximation of the control manifold $\bar{\mathcal{U}}$

Description of the method

We look for a *small* number of parameters $\nu \in \mathcal{K}$ approximating the manifold $\bar{\mathcal{U}}$ in the sense of the Kolmogorov width. **Roughly**, the Kolmogorov width measures how well we can approximate $\bar{\mathcal{U}}$ by a finite dimensional space.

With greedy algorithms (Cohen & DeVore, Volkwein, Buffa et. al, ...), we search for the **most representative values of $\bar{u}_\nu$**.

That is, given a tolerance ε, the goal is to find

$$\nu_1, \ldots, \nu_{n(\varepsilon)}$$

such that, for any other $\nu \in \mathcal{K}$, the corresponding control $\bar{u}_\nu$ can be approximated by $u^\star_\nu \in \mathrm{span}\{\bar{u}_{\nu_1}, \ldots, \bar{u}_{\nu_{n(\varepsilon)}}\}$ and

$$\|u^\star_\nu - \bar{u}_\nu\|_{L^2(\omega)} \leq \varepsilon.$$

We also want to minimize n.

The surrogate

In practical implementations, the set $\bar{\mathcal{U}}$ is unknown.

Given two parameters ν_1 and ν_2, how can we measure the distance between $\bar{u}_{\nu_1}$ and $\bar{u}_{\nu_2}$?

Recall that we want to avoid to compute $\bar{u}_\nu$.

The surrogate

In practical implementations, the set $\bar{\mathcal{U}}$ is unknown.

Given two parameters ν_1 and ν_2, how can we measure the distance between $\bar{u}_{\nu_1}$ and $\bar{u}_{\nu_2}$?

Recall that we want to avoid to compute $\bar{u}_\nu$.

Standard residual: Suppose that we have computed u_{ν_1}

$$|u_{\nu_1} - u_{\nu_2}| \sim \nabla J_{\nu_2}(u_{\nu_1}) - \nabla J_{\nu_2}(u_{\nu_2}) = \nabla J_{\nu_2}(u_{\nu_1})$$

Compute $\nabla J_{\nu_2}(u_{\nu_1}) = u_{\nu_1} + \beta S^*_{\nu_2}(S_{\nu_2} u_{\nu_1} - y_d)$, where S_ν is to control-to-state operator. This means

$$\begin{cases} -\text{div}(a_{\nu_2} \nabla y) + c\, y = \chi_\omega u_{\nu_1}, & \text{in } \Omega, \\ -\text{div}(a_{\nu_2} \nabla q) + c\, q = \beta\,(y - y^d), & \text{in } \Omega, \\ y = q = 0, & \text{on } \partial\Omega. \end{cases} \quad \Rightarrow -\chi_\omega q_{\nu_2} = S^*_{\nu_2}(S_{\nu_2} u_{\nu_1} - y_d)$$

Solving a cascade system.

Cheaper surrogates

A cheap surrogate: Instead of using $\bar{u}_\nu$ and approximate the manifold $\bar{\mathcal{U}}$, use the optimal variables $(\bar{q}_\nu, \bar{y}_\nu)$ and approximate the manifold $\bar{\mathcal{Q}} \times \bar{\mathcal{Y}}$.

Denoting $L_\nu z := -\text{div}(a_\nu \nabla z) + c\, z$, we define

$$R_\nu(q,y) := \begin{pmatrix} L_\nu y + \chi_\omega q \\ L_\nu q - \beta\,(y - y_d) \end{pmatrix} = \underbrace{G_\nu(q,y)}_{\text{linear part of} R_\nu(q,y)} + \begin{pmatrix} 0 \\ \beta y_d \end{pmatrix}.$$

With this definition, we are able to compute the following estimates:

$$c_1 \left(\|y - \bar{y}_\nu\|_{H_0^1 \Omega} + \|q - \bar{q}_\nu\|_{H_0^1(\Omega)} \right) \leq \|R_\nu(q,y)\|_{H^{-1}(\Omega)},$$

$$\|R_\nu(p,y)\|_{H^{-1}\Omega} \leq (1 + \alpha_2)(\|y - \bar{y}_\nu\|_{H_0^1(\Omega)} + \|q - \bar{q}_\nu\|_{H_0^1(\Omega)}).$$

where c_1 and α_2 only depending on a_1, a_2 and $\|c\|_\infty$.

Upper and lower bounds for $R_\nu(q,y)$ are essential for the proof of greedy algorithms in terms of the Kolmogorov width.

Main results

$$R_\nu(q, y) := \begin{pmatrix} L_\nu y + \chi_\omega q \\ L_\nu q - \beta\,(y - y_d) \end{pmatrix}. \tag{3}$$

Theorem 1 (H. Santmaria, L., Zuazua, '17)

The residual (3) provides the approximation estimates for optimal controls and states

- $\|u_\nu^\star - \bar{u}_\nu\|_{L^2(\Omega)} \leq \dfrac{1}{c_1}\|R_\nu(q, y)\|_{[H^{-1}(\Omega)]^2},$
- $\|y_\nu^\star - \bar{y}_\nu\|_{H_0^1(\Omega)} \leq \left(\dfrac{1}{\alpha_1 c_1}\right)\|R_\nu(q, y)\|_{[H^{-1}(\Omega)]^2},$

where c_1 and α_1 only depend on a_1, a_2 and $\|c\|_\infty$.

Offline algorithm

Step 1: *Initialization*. Fix $\varepsilon > 0$. Choose any $\nu \in \mathcal{K}$, $\nu = \nu_1$ and compute the minimizer of J_{ν_1}. This leads to $\begin{pmatrix} \bar{q}_{\nu_1} \\ \bar{y}_{\nu_1} \end{pmatrix}$.

Step 2: *recursive choice of ν*.

Assuming we have chosen $\nu_1, \ldots, \nu_p$, we choose ν_{p+1} as the maximizer of

$$\max_{\nu \in \mathcal{K}} \| \inf_{(q,y) \in (\bar{\mathcal{Q}}_p, \bar{\mathcal{Y}}_p)} R_\nu(q, y) \|$$

Step 3: *Stopping criterion*. Stop if the max $\leq \varepsilon$.

Theorem 1 (H-Santamaria, L., Zuazua, '17)

The offline algorithm stops after $n_0(\varepsilon)$ iterations, and fullfills the requirements of the greedy theory.

Online algorithm

After choosing the most representative values of ν, we can construct an approximated optimal control $u_\nu^\star$ for any arbitrary given value $\nu \in \mathcal{K}$ by taking

$$u_{\nu}^{\star} = \sum_{i=1}^{k} \lambda_i \bar{q}_{\nu_i}|_\omega$$

where λ_i are determined by the projection of the vector $\begin{pmatrix} 0 \\ y_d \end{pmatrix}$ to the space

$$\text{span}\{R_\nu(\bar{q}_{\nu_1}, \bar{y}_{\nu_1}), \ldots, R_\nu(\bar{q}_{\nu_k}, \bar{y}_{\nu_k})\}$$

Numerical results

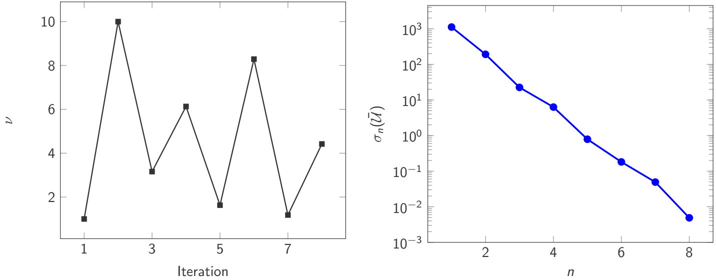
Numerical examples

- $\Omega = (0,1)^2$ in 2-D or $\Omega = (0,1)$ in 1-D.
- Uniform meshes, i.e., meshes with constant discretization steps in each direction, $N = 400$.
- We will approximate the operator $\mathcal{A} = -\text{div}(a(x,\nu)\nabla\,\cdot)$ by using the standard 5-point discretization.
- Discretize-then-optimize.
- $\nu \in \mathcal{K} = [1, 10]$.
- $\mathcal{K}$ sampled in 100 equidistant points.

Greedy test # 1

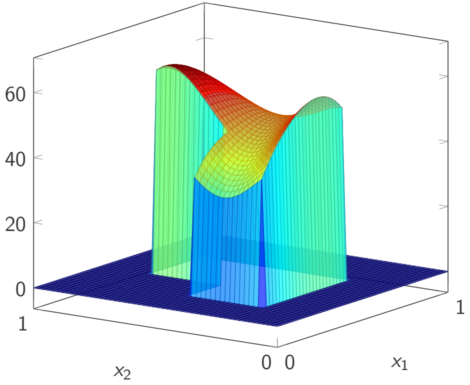
∘ $a(x,\nu) = 1 + \nu(x_1^2 + x_2^2)$, ∘ $c(x) = \sin(2\pi x_1)\sin(2\pi x_2)$,
∘ $y_d = sin(\pi x_1)$, ∘ $\beta = 10^4$, ∘ $\varepsilon = 0.005$

∘ $t_{\text{cheap}} = 304s$, ∘ $t_{std} = 384s$

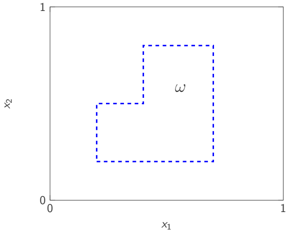


(a) Selected ν (b) Approximation error

Approximation for $\nu = \pi/2$



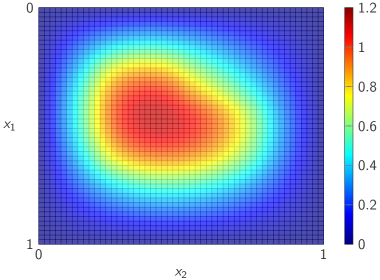
(c) The approximated control



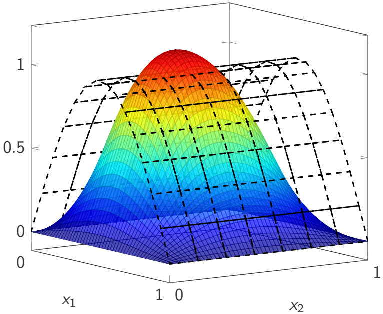
(d) The control set ω

$\circ\ |u^\star_{\pi/2} - \bar{u}_{\pi/2}|_{L^2(\omega)} \approx 1.45 \times 10^{-5}, \quad t_{\text{online}} = 0.45s, \quad t_{\text{iterative}} = 6.01s.$

Approximation for $\nu = \pi/2$ (cont.)



(e) The controlled state



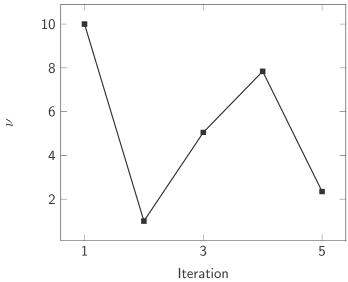
(f) The state $y^\star_{\pi/2}$ and the target function y^d (dashed)

○ $|y^\star_{\pi/2} - \bar{y}_{\pi/2}|_{L^2(\Omega)} \approx 1.15 \times 10^{-7}$

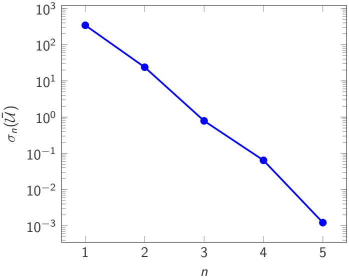
Greedy test # 2

$\circ\, a(x,\nu) = 1 + \nu x^2(1-x)^2$, $\circ\, c(x) = -15\sin(\pi x)$, $\circ\, y_d = \chi_{(0.5,0.8)}$, $\beta = 10^4$, $\circ\, \omega = (0.3, 0.9)$, $\circ\, \varepsilon = 0.005$

$\circ\, t_{\text{cheap}} = 0.68s$, $\circ\, t_{std} = 0.809s$

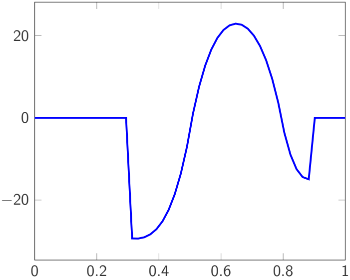


(g) Selected ν

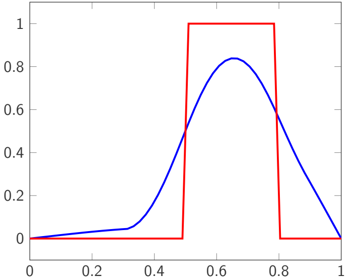


(h) Approximation error

Greedy test # 2



(i) The approximated control



(j) The state $y^{\star}_{\pi/2}$ (blue) and the target y^d (red)

$\circ\, |u^{\star}_{\pi/2} - \bar{u}_{\pi/2}|_{L^2(\omega)} \approx 1.17 \times 10^{-5}$, $\quad \circ\, \|y^{\star}_{\pi/2} - \bar{y}_{\pi/2}\|_{L^2(\Omega)} \approx 5.09 \times 10^{-7}$,

Connection with the turnpike problems

Time dependent control problem

Consider

$$\begin{cases} \partial_t y - \operatorname{div}(a(x,\nu)\nabla y) + c\,y = \chi_\omega u & \text{in } Q = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(x,0) = y^0(x) & \text{in } \Omega. \end{cases} \quad (4)$$

and the control problem

$$\min_u J_\nu{}^T(u) = \frac{1}{2}\int_0^T |u(t)|^2_{L^2(\omega)}dt + \frac{\beta}{2}\int_0^T \|y(t)-y^d\|^2_{L^2(\Omega)}dt.$$

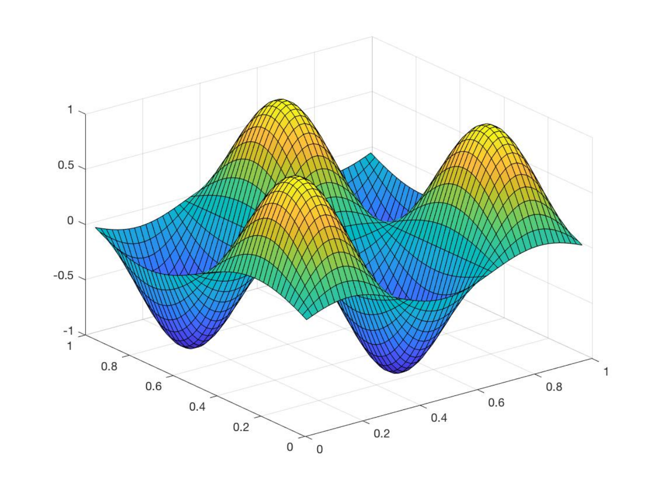
The optimal solution (u^T, y^T) satisfies

$$\|y^T(t)-\bar{y}\|_{L^2(\Omega)}+\|u^T(t)-\bar{u}\|_{L^2(\Omega)} \leq K\left(e^{-\mu t}+e^{-\mu(T-t)}\right), \quad \forall t \in [0,T]$$

- Exponential convergence of the finite-time horizon control problem to the steady one as $T \to \infty$.

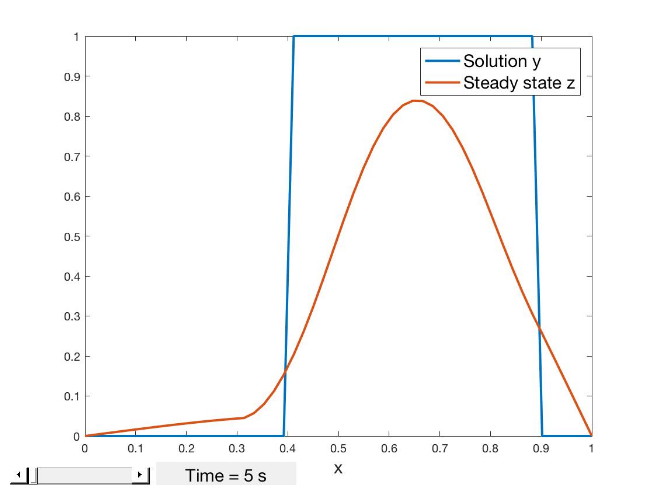
The case $c(x) \geq 0$ (greedy test #1)

$$u(x,t) = u^{\star}_{\pi/2}(x), \qquad y_0(x) = \sin(3\pi x_1)\sin(2\pi x_2)$$



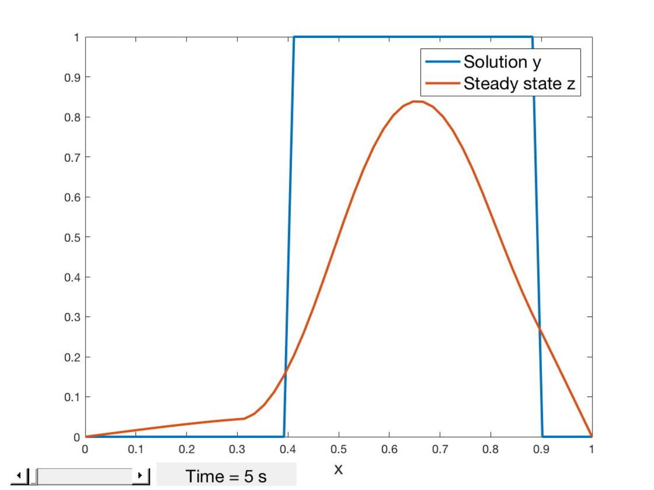
The case $c(x) \leq -\lambda_1$ (greedy test #2)

$u(x,t) = u^{\star}_{\pi/2}(x), \quad y_0(x) = \chi_{(0.4,0.9)}$



The case $c(x) \leq -\lambda_1$ (greedy test #2)

$u = u^T_{\pi/2}(x,t), \quad y_0(x) = \chi_{(0.4,0.9)}$



Thank you!