Sharp operator-norm asymptotics for linearised elastic plates with rapidly oscillating periodic properties

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Outline

- Resolvent norm convergence, Bloch formulation and Gelfand transform;
- Problem formulation;
- Important estimates;
- "Limit" equations;
- Some remarks about asymptotic procedure;
- Final conclusions.

Resolvent norm convergence, Bloch formulation and Gelfand transform

We look the problem

$$-\operatorname{div}(A(\frac{x}{\varepsilon})\nabla u) + u = f,$$

on $\mathbf{R}^n$, $f \in L^2(\mathbf{R}^n)$, $u \in H^1(\mathbf{R}^n)$. A is assumed to be one periodic symmetric, positive definite and bounded, i.e., there exist $\alpha, \beta > 0$

$$\alpha|\xi|^2 \leq A(x)\xi\cdot\xi \leq \beta|\xi|^2, \quad \forall x, \xi \in \mathbf{R}^n.$$

$$\mathcal{A}^\varepsilon u := -\operatorname{div} A(\frac{x}{\varepsilon})\nabla u.$$

Note that

$$(\mathcal{A}^\varepsilon + I)^{-1} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$$

is a continuous operator.

Resolvent norm convergence, Bloch formulation and Gelfand transform

The limit problem is given by (homogenization theory)

$$-\operatorname{div}(A^0\nabla u) + u = f,$$

$$A^0\xi\cdot\xi = \min_{\varphi\in H^1(\mathcal{Y})}\int_{\mathcal{Y}} A(y)(\xi+\nabla\varphi)\cdot(\xi+\nabla\varphi),$$

where $\mathcal{Y}$ is a flat torrus in $\mathbf{R}^n$.

$$\mathcal{A}^0 = -\operatorname{div}(A^0\nabla u).$$

Claim: If u^ε is a solution of ε-problem that corresponds to f^ε and if $f^\varepsilon \rightharpoonup f$ in L^2, then the $u^\varepsilon \rightharpoonup u$ in H^1, where u is a solution of zero problem.

The result can be quantified.

Resolvent norm convergence, Bloch formulation and Gelfand transform

In a series of papers Birman, Suslina proved:

$$\|(\mathcal{A}^\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1}\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \leq C\varepsilon,$$

$$\|(\mathcal{A}^\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1} - \varepsilon K(\varepsilon)\|_{L^2(\mathbf{R}^n)\to H^1(\mathbf{R}^n)} \leq C\varepsilon,$$

where $K(\varepsilon)$ is a corrector.
Later this was used for the estimates on finite domain by Suslina (2012), when one needs to include estimates of boundary layer (previous works with less sharp estimate by Zhikov, Pastukhova by Steklov smoothing, 2005, Griso by unfolding 2004).

Resolvent norm convergence, Bloch formulation and Gelfand transform

It can be shown that this is equivalent to

$$\|\exp(-t\mathcal{A}^{\varepsilon}) - \exp(-t\mathcal{A})\|\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \le C\frac{\varepsilon}{\sqrt{t}}.$$

This kind of estimates can be proven by looking corresponding parabolic problem and by spectral analysis (Zhikov & Pastukhova). There is also an interesting result of Ortega and Zuazua (2000) on approximation of operator potential $e^{-t\tilde{A}}$ as $t \to \infty$, where $\tilde{A}$ comes from a periodic problem.

Resolvent norm convergence, Bloch formulation and Gelfand transform

We explain the approach of Cherednichenko, Cooper (ARMA 2016) in the context of high contrast.

$$-\operatorname{div}(A(\frac{x}{\varepsilon})\nabla u) + u = f,$$

on $\mathbf{R}^n$, $f \in L^2(\mathbf{R}^n)$, $u \in H^1(\mathbf{R}^n)$. A is assumed to be 1-periodic:

$$A = \chi_1 A_1 + \chi_0 \varepsilon^2 A_0 \text{ on } \mathcal{Y},$$

where χ_0 is a characteristic function of e.g. ball $B \subset (0,1)^n$, $\chi_1 = 1 - \chi_0$, A_0, A_1 are symmetric, uniformily elliptic.
The qualitative analysis of these kind of operators was given by Zhikov. The limit operator is defined on the subspace of $L^2(\mathbf{R}^n \times \mathcal{Y})$ and its spectrum has band gap structure.

Resolvent norm convergence, Bloch formulation and Gelfand transform

Cherednichenko, Cooper found the operator $\tilde{\mathcal{A}}^\varepsilon$ which is simpler then the starting one and which satisfies

$$\|(\mathcal{A}^\varepsilon + I)^{-1} - (\tilde{\mathcal{A}}^\varepsilon + I)^{-1}\tilde{\mathcal{P}}^\varepsilon\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \leq C\varepsilon.$$

The operator $\tilde{\mathcal{A}}^\varepsilon$ is still ε- dependent and $\tilde{\mathcal{P}}^\varepsilon$ is a kind of projection. Resolvent approximation implies the approximation of spectrum of the operator $\mathcal{A}^\varepsilon$ in the Hausdorff sense. The operator $\tilde{\mathcal{A}}^\varepsilon$ differs from the limit operator obtained by qualitative analysis of Zhikov, since it contains more information. However, it is still computionally much cheaper than the original one. The method offers the way not only to prove the estimates, but also to change (or slightly perturb) the expected "limit" operator.

Resolvent norm convergence, Bloch formulation and Gelfand transform

$$Q' = [-\pi, \pi)^n, \quad Q = [0,1)^n.$$

We define the isometry $\mathcal{U}^\varepsilon : L^2(\mathbb{R}^n) \to L^2\big(\varepsilon^{-1}Q' \times Q\big)$ by

$$(\mathcal{U}^\varepsilon f)(\theta, y) = \left(\frac{\varepsilon^2}{2\pi}\right)^{n/2} \sum_{n \in \mathbf{Z}^n} f\big(\varepsilon(y+n)\big) \exp\big(-i\varepsilon\theta(y+n)\big),$$

$$\theta \in \varepsilon^{-1}Q', \quad y \in Q.$$

This isometry $\mathcal{U}^\varepsilon = \mathcal{G}^\varepsilon \mathcal{T}^\varepsilon$ is a composition of usual scaled Gelfand transform $\mathcal{G}^\varepsilon : L^2(\mathbb{R}^n) \to L^2\big(\varepsilon^{-1}Q' \times \varepsilon Q\big)$:

$$(\mathcal{G}^\varepsilon f)(\theta, z) = \left(\frac{\varepsilon}{2\pi}\right)^{n/2} \sum_{n \in \mathbf{Z}^n} f\big(z + \varepsilon n\big) \exp\big(-i\varepsilon\theta(y+n)\big), \quad z \in \varepsilon Q,$$

and the scaling transform $\mathcal{T}^\varepsilon : L^2(\varepsilon^{-1}Q' \times \varepsilon Q) \to L^2(\varepsilon^{-1}Q' \times Q)$

$$(\mathcal{T}^\varepsilon h)(\theta, y) = \varepsilon^{n/2} h(\theta, \varepsilon y).$$

Resolvent norm convergence, Bloch formulation and Gelfand transform

Notice that

$$((\mathcal{U}^{\varepsilon})^{-1}f)(x) = \left(\frac{\varepsilon^2}{2\pi}\right)^{-n/2} \int\limits_{\varepsilon^{-1}[-\pi,\pi)^2} f\left(\theta, \frac{x}{\varepsilon}\right) \exp\big(i\theta\cdot x\big) d\theta, \quad x \in \mathbb{R}^n.$$

Then we have that

$$\mathcal{U}^{\varepsilon}(\mathcal{A}^{\varepsilon} + I)^{-1}(\mathcal{U}^{\varepsilon})^{-1} = \int_{\varepsilon^{-1}Q'}^{\oplus} (\mathcal{B}_{\varepsilon,\theta} + I)^{-1} d\theta,$$

where $\mathcal{B}_{\varepsilon,\theta}$ is the operator generated by the sesquilinear form

$$b^{\varepsilon,\theta}(u,v) = \int_Q (\varepsilon^{-2}A_1 + A_0)(\nabla + i\varepsilon\theta)u \cdot \overline{(\nabla + i\varepsilon\theta)v}, \quad u, v \in H^1_{\#}(Q)$$

or

$$\tilde{b}^{\varepsilon,\theta}(u,v) = \int_Q (\varepsilon^{-2}A_1 + A_0)\nabla u \cdot \overline{\nabla v}, \quad u, v \in H^1_{\chi}(Q), \chi = \varepsilon\theta.$$

Resolvent norm convergence, Bloch formulation and Gelfand transform

For every parameter $\theta \in \varepsilon^{-1}Q'$ we obtain a differential equation on a compact domain Q (the equation can be looked with periodic or quasi-periodic boundary condition). This is a standard approach for periodic problems. In this way one can divide the spectrum of the original operator (on a non-compact domain) as a union of continuum family of spectrum of operators on a compact domain. One can even characterize generalized eigenfunctions of the original operator.
The novelty of the approach of Cherednichenko and Cooper consists in finding the operators $\mathcal{B}_{\mathrm{hom}}^{\varepsilon,\theta}$ such that

$$\|(\mathcal{B}^{\varepsilon,\theta} + I)^{-1} - (\mathcal{B}_{\mathrm{hom}}^{\varepsilon,\theta} + I)^{-1}\mathcal{P}^{\varepsilon}\|_{L^2(Q)\to L^2(Q)} \leq C\varepsilon,$$

where C is independent of θ and $\mathcal{P}^{\varepsilon}$ is a projection.

Resolvent norm convergence, Bloch formulation and Gelfand transform

The methodology consists in doing formal asymptotics for the solution of the equation

$$u_\theta^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n u_\theta^{(n)}, \quad u_\theta^{(n)} \in H^1_\#(Q).$$

plugging it into equation and obtaining the approximate solution. The difficulties arise in the fact that one has to do the estimates and the fact that there are two changing parameters (ε and θ) and that there are no rules (ansatz) how to do this asymptotics in the way to obtain the estimates. They had to analyze separately so called inner region ($|\theta| \leq 1$), intermediate region $1 \leq |\theta| \leq \varepsilon^{-1/2}$) and upper region $|\theta| \geq \varepsilon^{-1/2}$. In the case without high contrast upper region can be neglected, i.e., the good approximation of the solution is zero.

Problem formulation

$$\int_{\Omega^h} A\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \mathrm{sym}\nabla U^h : \mathrm{sym}\nabla \Phi^h + \rho^h \int_{\Omega^h} U^h \cdot \Phi^h = \int_{\Omega^h} F^h \cdot \Phi^h.$$

Here

$$\Omega^h = \mathbf{R}^2 \times (-h/2, h/2),\ U^h, \Phi^h \in H^1(\Omega^h, \mathbf{R}^3),\ F^h \in L^2(\Omega^h, \mathbf{R}^3).$$

A is one periodic, bounded and coercive on symmetric matrices, i.e., there exist $\alpha, \beta > 0$ such that

$$\alpha |M|^2 \le A(y_1, y_2) M \cdot M \le \beta |M|^2, \quad \forall (y_1, y_2) \in [0,1]^2, M \in \mathbf{R}^{3\times 3}_{\mathrm{sym}}.$$

Problem formulation

We transform the problem on the domain $\Omega = \mathbf{R}^2 \times I$, $I = (-\frac{1}{2}, \frac{1}{2})$ by doing change $x_3 = x_3^h/h$. We also do the scaling $F^h = (h^2 f_1, h^2 f_2, h^3 f_3)$, $U^h = (h^2 u_1, h^2 u_2, h^3 u_3)$, $\rho^h = h^2$. We also scale the test functions in the same way, do the Gelfand transform and scale in-plane components $y_\alpha = x_\alpha/\varepsilon$, for $\alpha = 1, 2$. After that we obtain:

$$\frac{1}{h^2}\int_Q A(y)\,\mathrm{sym}\left(\tilde{\nabla}^{\varepsilon,h,\theta}(hu_1, hu_2, u_3)\right) : \overline{\mathrm{sym}\left(\tilde{\nabla}^{\varepsilon,h,\theta}(h\varphi_1, h\varphi_2, \varphi_3)\right)}$$
$$+h^2\int_Q u_\alpha\overline{\varphi}_\alpha + \int_Q u_3\overline{\varphi}_3 = \int_Q f_i\,\overline{\varphi}_i, \quad \forall\varphi \in H^1_\#(Q, \mathbf{C}^3).$$

Here

$$Q = Q_r \times I,\ Q_r = (0,1)^2,\ \theta \in \varepsilon^{-1}(-\pi, \pi)^2.$$

$$(\tilde{\nabla}^{\varepsilon,h,\chi} v)_{i\alpha} := \frac{1}{\varepsilon}(\partial_\alpha + i\chi_\alpha)v_i,$$

$$(\tilde{\nabla}^{\varepsilon,h,\chi} v)_{i3} := \frac{1}{h}\partial_3 v_i, \quad \alpha = 1, 2,\ i = 1, 2, 3.$$

Problem formulation

We will look the regime $\varepsilon = h$. The equations suggest that we take $\tilde{u}_\alpha = \varepsilon u_\alpha$. We obtain:

$$\frac{1}{\varepsilon^4}\int_Q A(y)\,\mathrm{sym}\tilde{\nabla}_{y,x_3}(\widetilde{u}_1,\widetilde{u}_2,u_3) : \overline{\mathrm{sym}\tilde{\nabla}_{y,x_3}(\varphi_1,\varphi_2,\varphi_3)}$$

$$+\int_Q \widetilde{u}_\alpha\overline{\varphi_\alpha} + \int_Q u_3\overline{\varphi_3} = \frac{1}{\varepsilon}\int_Q f_\alpha\overline{\varphi_\alpha} + \int_Q f_3\overline{\varphi_3}, \quad \forall\varphi\in H^1_\chi(Q,\mathbf{C}^3).$$

Here

$$(\tilde{\nabla}u)_{i\alpha} = \partial_\alpha u_i + i\chi_\alpha u_i,\ (\tilde{\nabla}u)_{i3} = \partial_3 u_i,\ \alpha = 1,2,\ i = 1,2,3.$$

This change means that we have to identify the in-plane components up to order ε^2 in L^2 norm. The equations can be looked on the space of χ- quasiperiodic functions $H^1_\chi(Q,\mathbf{C}^3)$ in which case we replace $\tilde{\nabla}$ by ∇.

Important estimates-Korn type inequalities

By using Korn's inequality and boundary condition we can show that for $u \in H^1_\chi(Q, \mathbf{C}^3)$ we have

$$
\begin{aligned}
\left\| u_1 - (c_1 - i\chi_1 c_3 x_3) \exp\big(i(\chi, y)\big) \right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
\left\| u_2 - (c_2 - i\chi_2 c_3 x_3) \exp\big(i(\chi, y)\big) \right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
\left\| u_3 - c_3 \exp\big(i(\chi, y)\big) \right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)},
\end{aligned}
$$

for some $c_1, c_2, c_3 \in \mathbf{C}$ which satisfy

$$
\begin{aligned}
\max\{|c_1|, |c_2|\} &\lesssim \min\left\{\frac{1}{|\chi_1|}, \frac{1}{|\chi_2|}\right\} \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
|c_3| &\lesssim \min\left\{\frac{1}{|\chi_1|^2}, \frac{1}{|\chi_2|^2}\right\} \|\mathrm{sym}\nabla u\|_{L^2(Q)}
\end{aligned}
$$

The following is satisfied

$$
\begin{aligned}
\chi \neq 0, \mathrm{sym}\,\nabla u = 0 \quad &\Longrightarrow \quad u = 0, \\
\chi = 0, \mathrm{sym}\,\nabla u = 0 \quad &\Longrightarrow \quad u = Ax + b,\ A \in \mathbf{C}^{3\times 3}_{\mathrm{skew}}, b \in \mathbf{C}^3.
\end{aligned}
$$

Important estimates-apriori estimates intermediate region

We look intermediate region $|\theta| \geq 1$ and introduce

$$D(\theta) := \max\left\{\min\left\{\frac{1}{|\theta_1|}, \frac{1}{|\theta_2|}\right\}, \min\left\{\frac{1}{|\theta_1|^2}, \frac{1}{|\theta_2|^2}\right\}\right\}.$$

The solution $u = (\tilde{u}_1, \tilde{u}_2, u_3) \in H^1_\chi(Q, \mathbf{C}^3)$ satisfies

$$\begin{aligned}
\left\|\widetilde{u}_1 - (c_1 - i\chi_1 c_3 x_3)\exp\big(i(\chi, y)\big)\right\|_{H^1(Q)} &\lesssim D(\theta)\varepsilon^2\|f\|_{L^2(Q)},\\
\left\|\widetilde{u}_2 - (c_2 - i\chi_2 c_3 x_3)\exp\big(i(\chi, y)\big)\right\|_{H^1(Q)} &\lesssim D(\theta)\varepsilon^2\|f\|_{L^2(Q)},\\
\left\|u_3 - c_3\exp\big(i(\chi, y)\big)\right\|_{H^1(Q)} &\lesssim D(\theta)\varepsilon^2\|f\|_{L^2(Q)}.,
\end{aligned}$$

where

$$\begin{aligned}
\max\{|c_1|, |c_2|\} &\lesssim D(\theta)\min\left\{\frac{1}{|\theta_1|}, \frac{1}{|\theta_2|}\right\}\varepsilon\|f\|_{L^2(Q)},\\
|c_3| &\lesssim D(\theta)\min\left\{\frac{1}{|\theta_1|^2}, \frac{1}{|\theta_2|^2}\right\}\|f\|_{L^2(Q)}.
\end{aligned}$$

Important estimates-apriori estimates

Comments:

- In the upper region to obtain the solution of the appropriate precision it is enough to find some equation for that is satisfied by $c_1, c_2, c_3 \in \mathbf{C}$ that satisfy the above approximation;
- These estimates imply that for $|\theta| \geq \varepsilon^{-1/2}$ we can take $c_1 = c_2 = c_3 = 0$. This does not happen in high contrast case (Cherednichenko, Cooper);
- Inner region is more complex in the asymptotics since it contains singularity. In order to deal with these problems easier we divide the problem in three cases. We will additionally have to assume some planar symmetries of the elastic tensor. Moreover it will be seen that we will need more from the solution than identifying just three constants. This is very unusual for the homogenization problems;
- Apriori estimates are not used in the asymptotic procedure. They just serve as a help to guess what one should obtain in the "limit".

Important estimates-apriori estimates inner region, vertical forces

We additionally assume "planar" material symmetries:

$$A_{\alpha\beta\gamma 3} = 0, \qquad A_{\alpha 3333} = 0 \qquad \forall \alpha, \beta, \gamma \in \{1, 2\}.$$

If we take $f_1 = f_2 = 0$ we have the following estimates

$$\begin{aligned}
\left\| \widetilde{u}_1 - (c_1 - i\chi_1 c_3 x_3) \exp\big(i(\chi, y)\big) \right\|_{H^1(Q)} &\lesssim \varepsilon^2 \|f_3\|_{L^2(Q)}, \\
\left\| \widetilde{u}_2 - (c_2 - i\chi_2 c_3 x_3) \exp\big(i(\chi, y)\big) \right\|_{H^1(Q)} &\lesssim \varepsilon^2 \|f_3\|_{L^2(Q)}, \\
\left\| u_3 - c_3 \exp\big(i(\chi, y)\big) \right\|_{H^1(Q)} &\leq \varepsilon^2 \|f_3\|_{L^2(Q)}, \\
\max\big\{|c_1|, |c_2|, |c_3|\big\} &\lesssim \|f_3\|_{L^2(Q)}.
\end{aligned}$$

In the case of vertical forces in the inner region one should not expect difficulties.

Important estimates-apriori estimates inner region, horizontal forces

The case of horizontal forces $f_3 = 0$ we divide into two subcases: forces odd in x_3 variable and forces even in x_3 variable. For the odd forces we have the following estimates

$$\begin{aligned} \left\| \widetilde{u}_1 + i\chi_1 c_3 x_3 \exp(i(\chi, y)) \right\|_{H^1(Q)} &\lesssim \varepsilon^2 \left\| (f_1, f_2) \right\|_{L^2(Q)}, \\ \left\| \widetilde{u}_2 + i\chi_2 c_3 x_3 \exp(i(\chi, y)) \right\|_{H^1(Q)} &\lesssim \varepsilon^2 \left\| (f_1, f_2) \right\|_{L^2(Q)}, \\ \left\| u_3 - c_3 \exp(i(\chi, y)) \right\|_{H^1(Q)} &\lesssim \varepsilon^2 \left\| (f_1, f_2) \right\|_{L^2(Q)}, \\ |c_3| &\lesssim \left\| (f_1, f_2) \right\|_{L^2(Q)}. \end{aligned}$$

Planar symmetries of the elasticity tensor are used to conclude $c_1 = 0, c_2 = 0$.

Important estimates-apriori estimates inner region, horizontal forces

For even planar forces we have the estimates

$$\begin{aligned}
\left\|\widetilde{u}_1 - c_1 \exp(i(\chi, y))\right\|_{H^1(Q)} &\lesssim \varepsilon \|f\|_{L^2(Q)}, \\
\left\|\widetilde{u}_2 - c_2 \exp(i(\chi, y))\right\|_{H^1(Q)} &\lesssim \varepsilon \|f\|_{L^2(Q)}, \\
\left\|u_3 - c_3 \exp(i(\chi, y))\right\|_{H^1(Q)} &\lesssim \varepsilon \|f\|_{L^2(Q)}, \\
\max\{|c_1|, |c_2|\} &\lesssim \varepsilon^{-1} \|f\|_{L^2(Q)}.
\end{aligned}$$

Planar symmetries of the elasticity tensor are used to conclude $c_3 = 0$. This is the problematic part of the inner region since in this case to obtain the estimates it is not enough to identify c_1, c_2, but we need to go further in the expansion. Moreover c_1, c_2 are of order ε^{-1}.

Important estimates for the asymptotic expansion

We introduce the space

$$F(\chi) := \left\{ \begin{pmatrix} c_1 - i\chi_1 c_3 x_3 \\ c_2 - i\chi_2 c_3 x_3 \\ c_3 \end{pmatrix} \exp\big(i(\chi, y)\big) : \; c_1, c_2, c_3 \in \mathbf{C} \right\}$$

For $u \in H^1_\chi(Q, \mathbf{C}^3)$, that satisfies

$$\int_Q A(y)\, \mathrm{sym}\nabla u : \mathrm{sym}\nabla v = \int_Q f \cdot v \quad \forall v \in H^1_\chi(Q, \mathbf{C}^3),$$

we have the following estimates: If $f \in F(\chi)^\perp$ we have

$$\|u_1\|_{H^1(Q)} \lesssim \min\left\{ \frac{1}{|\chi_1|}, \frac{1}{|\chi_2|} \right\} \|f\|_{H^{-1}(Q)},$$

$$\|u_2\|_{H^1(Q)} \lesssim \min\left\{ \frac{1}{|\chi_1|}, \frac{1}{|\chi_2|} \right\} \|f\|_{H^{-1}(Q)},$$

$$\|u_3\|_{H^1(Q)} \lesssim \min\left\{ \frac{1}{|\chi_1|^2}, \frac{1}{|\chi_2|^2} \right\} \|f\|_{H^{-1}(Q)},$$

Important estimates for the asymptotic expansion

For general f we have

$$
\begin{aligned}
\|u_1\|_{H^1(Q)} &\lesssim \min\left\{\frac{1}{|\chi_1|^2}, \frac{1}{|\chi_2|^2}\right\} \|(f_1, f_2)\|_{H^{-1}(Q)} \\
&\quad + \min\left\{\frac{1}{|\chi_1|^3}, \frac{1}{|\chi_2|^3}\right\} \|f_3\|_{H^{-1}(Q)}, \\
\|u_2\|_{H^1(Q)} &\lesssim \min\left\{\frac{1}{|\chi_1|^2}, \frac{1}{|\chi_2|^2}\right\} \|(f_1, f_2)\|_{H^{-1}(Q)} \\
&\quad + \min\left\{\frac{1}{|\chi_1|^3}, \frac{1}{|\chi_2|^3}\right\} \|f_3\|_{H^{-1}(Q)}, \\
\|u_3\|_{H^1(Q)} &\lesssim \min\left\{\frac{1}{|\chi_1|^3}, \frac{1}{|\chi_2|^3}\right\} \|(f_1, f_2)\|_{H^{-1}(Q)} \\
&\quad + \min\left\{\frac{1}{|\chi_1|^4}, \frac{1}{|\chi_2|^4}\right\} \|f_3\|_{H^{-1}(Q)}.
\end{aligned}
$$

"Limit" equation

Ciarlet and Kesavan (1981) looked the spectral problem on the bounded domain $\Omega = \omega \times I$, where $\omega \subset \mathbf{R}^2$ is open bounded set with Lipschitz boundary in the case of isotropic homogeneous plate (clamped plate). In the limit they obtained the folowing problem: find $u_3 \in H^2(\omega, \mathbf{R}^3)$, $u_3 = \partial_n u_3 = 0$ on $\partial\omega$ that satisfies

$$\int_\omega \left(\frac{4\lambda\mu}{3(\lambda + 2\mu)} \Delta u_3 \Delta v_3 + \frac{4\mu}{3} \partial_{\alpha\beta} u_3 \partial_{\alpha\beta} v_3 \right) = 2\Lambda \int_\omega u_3 v_3,$$

$$v_3 \in H^2(\omega, \mathbf{R}^3), v_3 = \partial_n v_3 = 0 \text{ on } \partial\omega.$$

"Limit" equation

Comments:

- The limit equation is the spectral problem of forth order for the vertical displacement. It is proved that eigenvalues of the original problem converge to the eigenvalues of this limit problem in the Hausdorff sense;
- The fact that we obtain the limit problem only for u_3 is the consequence of scaling for the density and the fact that in the limit problem (in the case of isotropic media) the equations for the vertical displacement separate from the equations of horizontal displacements;
- Ciarlet scaling is the only reasonable scaling for the bounded domain;
- In the case when we have non-homogeneous elasticity tensor in x_3 direction the separation of horizontal and vertical displacement does not happen.
- For eigenfunctions the limit horizontal displacement is $u_1 = -x_3 \partial_1 u_3$, $u_2 = -x_3 \partial_2 u_3$.

"Limit" equation

We look for the solution $(m_1, m_2, m_3) \in \mathbf{C}^3$ to the identity

$$
\begin{aligned}
& A^{\text{hom}}(m_1, m_2, m_3)^\top \cdot \overline{(d_1, d_2, d_3)^\top} \\
& + \varepsilon^4 \Bigg(\int_Q (m_1 - i\chi_1 x_3 m_3)(\overline{d_1 - i\chi_1 x_3 d_3}) \\
& + \int_Q (m_2 - i\chi_2 x_3 m_3)(\overline{d_2 - i\chi_2 x_3 d_3}) + \int_Q m_3 \overline{d_3} \Bigg) \\
= \ & \varepsilon^3 \int_Q f_1 (\overline{d_1 - i x_3 \chi_1 d_3}) + \varepsilon^3 \int_Q f_2 (\overline{d_2 - i x_3 \chi_2 d_3}) + \varepsilon^4 \int_Q f_3 \overline{d_3}
\end{aligned}
$$

where for (m_1, m_2, m_3), $(d_1, d_2, d_3) \in \mathbf{C}^3$ we define

$$
\begin{aligned}
& A^{\text{hom}}(m_1, m_2, m_3)^\top \cdot \overline{(d_1, d_2, d_3)^\top} \\
& := \int_Q A(y) \big(\nabla w_m + K(\chi, m_1, m_2) - i x_3 L(\chi, m_3) \big) : \\
& \qquad \big(\overline{K(\chi, d_1, d_2) - i x_3 L(\chi, d_3)} \big),
\end{aligned}
$$

"Limit" equation

where

$$K(\chi, m_1, m_2) := i \begin{bmatrix} \chi_1 m_1 & \frac{1}{2}(\chi_1 m_2 + \chi_2 m_1) \\ \frac{1}{2}(\chi_1 m_2 + \chi_2 m_1) & \chi_2 m_2 \end{bmatrix},$$

$$L(\chi, m_3) := i m_3 \begin{bmatrix} \chi_1^2 & \chi_1 \chi_2 \\ \chi_1 \chi_2 & \chi_2^2 \end{bmatrix}.$$

The corrector $w_m \in H^1_\#(Q, \mathbf{C}^3)$ satisfies

$$(\mathrm{sym}\,\nabla)^* \cdot A(y) \nabla w_m = -(\mathrm{sym}\,\nabla)^* \cdot A(y)\big(K(\chi, m_1, m_2) - i x_3 L(\chi, m_3)\big).$$

Some remarks about asymptotic procedure

$$\frac{1}{\varepsilon^4}\left(\mathrm{sym}\tilde{\nabla}_{y,x_3}\right)^* \cdot A(y)\mathrm{sym}\tilde{\nabla}_{y,x_3}\tilde{u} + \tilde{u} \quad = \quad f^\varepsilon$$

$$\tilde{u} = (\tilde{u}_1, \tilde{u}_2, u_3),\ f^\varepsilon = (\frac{1}{\varepsilon}f_1, \frac{1}{\varepsilon}f_2, f_3).$$

In the asymptotic procedure we start from the solution of the "limit" equation

- In the intermediate region we start from

$$u^0 = (m_1 - i\chi_1 x_3 m_3, m_2 - i\chi_2 x_3 m_3, m_3)^\top;$$

- In the inner region in the case of vertical forces or in-plane forces that are odd in x_3 variable we start form

$$u^0 = (-i\chi_1 x_3 m_3, -i\chi_2 x_3 m_3, m_3)^\top;$$

- In the inner region in the case in-plane forces that are even in x_3 variable we start form

$$u^0 = (m_1, m_2, 0)^\top;$$

Some remarks about asymptotic procedure

- We look for the approximate solution $z = u^0 + u^1 + u^2 + \dots$ that satisfies

$$\frac{1}{\varepsilon^4}\left(\mathrm{sym}\tilde{\nabla}_{y,x_3}\right)^* \cdot A(y)\mathrm{sym}\tilde{\nabla}_{y,x_3} z + z = f^\varepsilon + \mathrm{Error},$$

  where

$$\|\mathrm{Error}\|_{H^{-1}(Q)} \lesssim \max\{\varepsilon^6, \varepsilon^5|\theta|^4\}\|f\|_{L^2(Q)}.$$

  Then it can be shown by subtraction and estimates that

$$\|\tilde{u}_\alpha - z_\alpha\|_{H^1(Q)} \lesssim \varepsilon^2\|f\|_{L^2(Q)},\ \|u_3 - z_3\|_{H^1(Q)} \lesssim \varepsilon\|f\|_{L^2(Q)};$$

- The difficulties in the asymptotic procedure arise from the fact that we have to go very high in precision;
- In the intermediate region and inner region in the case of vertical forces and odd in-plane forces we can use the estimates on $u^1, u^2, \dots$ that tell us that we can take $z = u^0$;

Some remarks about asymptotic procedure

- In the inner region in the case of even in-plane forces we need to update u^0 by adding u^1 and $\tilde{u}^1 = C$

$$u^1 = i\varepsilon\theta_1 m_1 \varphi^1 + \frac{i}{2}(\varepsilon\theta_2 m_1 + i\varepsilon\theta_1 m_2)\varphi^2 + i\varepsilon\theta_2 m_2 \varphi^3,$$

where the functions φ^i, $i = 1, 2, 3$, satisfy

$$(\mathrm{sym}\,\nabla)^* \cdot A(y)\,\mathrm{sym}\,\nabla\varphi^i = -(\mathrm{sym}\,\nabla)^* \cdot A(y) M_i^1, \quad \int_Q \varphi^i = 0.$$

Here

$$M_1^1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M_2^1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad M_3^1 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};$$

Some remarks about asymptotic procedure

- The constant $C = (C_1, C_2, 0)^\top$ (depending on θ) satisfies

$$\int_Q A(y)\big(\mathrm{sym}\,\nabla u^2 + (i\varepsilon\theta)(u^1 + C)\big) : \overline{\big(\mathrm{sym}\,\nabla\psi + (i\varepsilon\theta)D\big)}$$

$$+ \varepsilon^4 C \cdot \overline{D}$$

$$= -\int_Q A(y)\big(\mathrm{sym}\,\nabla u^1 + K(\chi, m_1, m_2)\big) : \overline{(i\varepsilon\theta)\psi}$$

$$- \varepsilon^4 \int_Q (m_1, m_2, 0)^\top \cdot \overline{\psi} + \varepsilon^3 \int_Q (f_1, f_2, 0)^\top \cdot \overline{\psi}$$

$$\forall\psi \in H^1_\#(Q, \mathbb{R}^3),\ D = (D_1, D_2, 0)^\top, \qquad \int_Q u^2 = 0;$$

- Notice that in the case when the elasticity tensor is constant (independent of y) we have that $u^1 = 0$, $C = 0$. Thus the appearence of these terms in the approximate problem is only the consequence of inhomogeneties;

Final conclusions

- The "limit" problem in the case $f_1 = f_2 = 0$ and under the assumption of planar symmetries is

$$\frac{1}{12}\int_{\mathbf{R}^2} A^0\nabla^2 v : \nabla^2\psi + \frac{\varepsilon^2}{12}\int_{\mathbf{R}^2} \nabla v \cdot \nabla\psi + \int_{\mathbf{R}^2} v\psi$$
$$= \int_{\mathbf{R}^2} (\mathcal{C}^\varepsilon f_3)\,\psi, \quad \forall\psi \in H^2(\mathbf{R}^2),$$

  where A^0 is the fourth-order symmetric tensor given by

$$A^0 M : M = \min_{\psi\in H^1_\#(Q,\mathbf{R}^3)} \int_Q A(y)\big(M+\mathrm{sym}\,\nabla\psi\big) : \big(M+\mathrm{sym}\,\nabla\psi\big),$$

  and $\mathcal{C}^\varepsilon : L^2(\mathbf{R}^2 \times I) \to L^2(\mathbf{R}^2)$ is given by $\mathcal{C}^\varepsilon = (\mathcal{U}^\varepsilon)^{-1}\mathcal{P}^\varepsilon\mathcal{U}^\varepsilon$, where $\mathcal{U}^\varepsilon : L^2(\mathbf{R}^2 \times I) \to L^2\big(\varepsilon^{-1}[-\pi,\pi)^2 \times Q\big)$ is the Gelfand transform (only in x_1, x_2) and $\mathcal{P}^\varepsilon : L^2\big(\varepsilon^{-1}[-\pi,\pi)^2 \times Q\big) \to L^2\big(\varepsilon^{-1}[-\pi,\pi)^2\big)$ are given by

$$(\mathcal{P}^\varepsilon f)(\theta) = \int_Q f(\theta,\cdot,\cdot);$$

Final conclusions

- In the case of horizontal forces that are even in the x_3-variable the "limit" problem is

$$\int_{\mathbf{R}^2} A^0 \,\mathrm{sym}\,\nabla u : \mathrm{sym}\,\nabla\psi + \varepsilon^2 \int_{\mathbf{R}^2} u \cdot \psi$$
$$= \int_{\mathbf{R}^2} (\mathcal{C}^\varepsilon f_1)\,\psi_1 + \int_{\mathbf{R}^2} (\mathcal{C}^\varepsilon f_2)\,\psi_2,\ \forall\psi \in H^1(\mathbf{R}^2, \mathbf{R}^2),$$

  whose solution is corrected by adding the expression

$$\partial_1 u_1 \varphi_1\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, x_3\right) + \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1)\varphi_2\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, x_3\right)$$
$$+\partial_2 u_2 \varphi_3\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, x_3\right) + \mathcal{U}_\varepsilon^{-1} C^\varepsilon(\theta),$$

  where φ_i, $i = 1, 2, 3$, and $C^\varepsilon(\theta)$ satisfy the above equations;

Final conclusions

- In the case of horizontal forces that are odd in the x_3-variable, the "limit" problem is given by

$$\frac{1}{12}\int_{\mathbb{R}^2} A^0\nabla^2 v : \nabla^2\psi + \frac{\varepsilon^2}{12}\int_{\mathbf{R}^2}\nabla v\cdot\nabla\psi + \int_{\mathbf{R}^2} v\psi$$
$$= -\int_{\mathbf{R}^2}\left(\widetilde{\mathcal{C}}^\varepsilon f_1\right)\partial_1\psi - \int_{\mathbf{R}^2}\left(\widetilde{\mathcal{C}}^\varepsilon f_2\right)\partial_2\psi \quad \forall\psi\in H^2(\mathbf{R}^2),$$

where $\widetilde{\mathcal{C}}^\varepsilon = (\mathcal{U}^\varepsilon)^{-1}\widetilde{\mathcal{P}}^\varepsilon\mathcal{U}^\varepsilon$, and the operators $\widetilde{\mathcal{P}}^\varepsilon : L^2\big(\varepsilon^{-1}[-\pi,\pi)^2\times Q\big) \to L^2\big(\varepsilon^{-1}[-\pi,\pi)^2\big)$ are given by

$$(\widetilde{\mathcal{P}}^\varepsilon f)(\theta) = \int_Q x_3 f(\theta,\cdot,\cdot);$$

Final conclusions

- The complete solution is the superposition of these three casses since
$$f(\cdot, x_3) = \underbrace{\frac{f(\cdot, x_3) + f(\cdot, -x_3)}{2}}_{\text{even}} + \underbrace{\frac{f(\cdot, x_3) - f(\cdot, -x_3)}{2}}_{\text{odd}};$$
- The limit equation are of the type of Ciarlet's plate on infinite domain with lower order term included and updated term in the case of in-plane even force;
- The scaling that we imposed inluences the model significantly. Notice that this scaling is not "suggested by the equations" since the solution does not stay bounded in $L^2(\mathbf{R}^2)$ as $\varepsilon \to 0$. However, it is the only reasonable scaling on the finite domain. Other "more natural" scaling would not cause that the limit model is Ciarlet's plate;
- One can use the obtained equations to calculate the approximate spectral density of the original operator (the computations are done on transformed problem).

Thank you for your attention!