Distributed optimal control of parabolic equations by spectral decomposition

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The problem framework

The constrained minimisation problem

$$(\mathcal{P}) \qquad \min_u \left\{ J(u) : \quad y(T) \in \overline{B_\varepsilon\left(y^T\right)} \right\},$$

where:
– J is a given cost functional
– y^T is a given target
– y the solution of

$$(\mathcal{E}) \qquad \begin{cases} \frac{d}{dt} y(t) + \mathcal{A} y(t) = \mathcal{B}_t u(t) & \text{for } t \in (0, T) \\ y(0) = 0. \end{cases} \tag{1}$$

H1 The functional J is strictly convex, coercive and lower-semicontinuous.

H2 The unbounded linear operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is positive semidefinite, selfadjoint with dense domain $D(\mathcal{A})$ and compact resolvent.

H3 The operator $\mathcal{B}_t$ belongs to $\mathcal{L}\left(\mathcal{U}, \mathcal{H}\right)$ for each time $t \in (0, T)$; moreover the pair (A, B_t) is approximately controllable in time T.

U, H - real Hilbert space

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The main example

Heat equation:

$$\begin{cases} \frac{d}{dt}y(t) - \Delta y(t) = \mathbb{1}_\omega u(t) & \text{in } \Omega \times (0,T) \\ y(t) = 0 & \text{on } \partial\Omega \times (0,T) \\ y(0) = 0 & \text{in } \Omega. \end{cases} \tag{2}$$

Functional:

$$J(u) = \frac{1}{2}\int_0^T \alpha(t)\ \|u(t)\|^2_{\mathcal{U}}\ dt\ +\ \frac{1}{2}\int_0^T \beta(t)\ \|y_u(t) - y^d(t)\|^2_{\mathcal{H}}\ dt.$$

The system (2) is not exactly controllable.
For any open subset ω of positive measure system (2) is approximately controllable in any time $T > 0$.
The goal: among all the eligible controls to detect one minimising given cost functional.

Existence of the solution

Unconstrained problem:

$$\tilde{u} = \arg \min_{u \in L^2_{T,\mathcal{U}}} J(u). \tag{3}$$

It admits the unique solution $\tilde{u}$ (due to assumptions on J).

Theorem

The constrained problem $(\mathcal{P})$ admits a unique solution that we denote by $\hat{u}$.

If $\|\tilde{y}(T) - y^T\| \leq \varepsilon$, then the optimal control coincides with the solution of the unconstrained problem, i.e. $\hat{u} = \tilde{u}$.

Otherwise, the optimal final state verifies $\|\hat{y}\,(T) - y^T\|_{\mathcal{H}} = \varepsilon$ (i.e.: $\hat{y}\,(T)$ lies on $\partial B_\varepsilon\left(y^T\right)$).

In the sequel we suppose that $\varepsilon < \|\tilde{y}(T) - y^T\|$.

Characterisation of the solution by the dual problem

We introduce the Fenchel conjugate $J^\star$ of the functional J:

$$J^\star\left(u^\star\right) = \sup_{u \in L^2_{T,\mathcal{U}}} \left\{ \langle u^\star, u\rangle_{T,\mathcal{U}} - J(u) \right\} \quad \text{for } u^\star \in L^2_{T,\mathcal{U}}.$$

Theorem [Generalized HUM]

Let $\bar{y} \in \mathcal{H}$ be a reachable state.
Then

$$\bar{u} \in \arg\min_{u \in \mathcal{U}} \left\{ J(u) : \quad \mathcal{T}u = \bar{y} \right\}. \tag{4}$$

is of the form $\bar{u} = \nabla J^\star\left(-\mathcal{T}^*\bar{\varphi}^T\right)$, where

$$\bar{\varphi}^T \in \arg\min_{\varphi^T \in \mathcal{H}} \left\{ J^\star(-\mathcal{T}^*\varphi^T) + \langle \bar{y}, \varphi^T\rangle_{\mathcal{H}} \right\}. \tag{5}$$

$\mathcal{T} : L^2_{T,\mathcal{U}} \to \mathcal{H}$ is the operator that takes the distributed control and gives the corresponding final state

$$\mathcal{T}u = y(T).$$

$$\mathcal{T}^*\varphi^T = \mathcal{B}^*\varphi,$$

where φ is the solution to the dual problem satisfying $\varphi(T) = \varphi^T$.

Characterisation of the solution by the dual problem

It is enough to restrict minimisation problem $(\mathcal{P})$ to controls of form $u = \nabla J^{\star}\left(-\mathcal{T}^{*}\varphi^{T}\right)$.
For such u

$$J(u) = F(\varphi^{T}),$$

where

$$F(\varphi^{T}) = -\left[\langle \nabla J^{\star}\left(-\mathcal{T}^{*}\varphi^{T}\right),\, \mathcal{T}^{*}\varphi^{T}\rangle_{L^{2}_{T,\mathcal{U}}} + J^{\star}\left(-\mathcal{T}^{*}\varphi^{T}\right)\right]. \quad (6)$$

Theorem

The solution of problem $(\mathcal{P})$ is

$$\hat{u} = \nabla J^{\star}\left(-\mathcal{T}^{*}\hat{\varphi}^{T}\right),$$

where $\hat{\varphi}^{T}$ is a solution of

$$\min_{\varphi^{T}\in\mathcal{H}} \left\{ F(\varphi^{T}) : \quad \|y(T) - y^{T}\|_{\mathcal{H}} = \varepsilon. \right\}. \quad (7)$$

Quadratic cost-functional

$$J(u) = \frac{1}{2}\|Cu - d\|^2_{\mathcal{X}}, \tag{8}$$

C – a linear bounded operator from $L^2_{T,\mathcal{U}}$ to a generic Hilbert space $\mathcal{X}$
We suppose that C is *uniformly elliptic*:

$$\|Cu\|_{\mathcal{X}} \geq \gamma \|u\|_{L^2_{T,\mathcal{U}}}. \tag{9}$$

It implies that it exists $(C^*C)^{-1}$.

EXAMPLE

Set $C = (C_1, C_2)$ and $d = (d_1, d_2)$:

$$(C_1 u)(t) = \sqrt{\alpha(t)}\; u(t)\; \mathbb{1}_{\omega}; \tag{10}$$

$$(C_2 u)(t) = \sqrt{\beta(t)}\; y_u(t)\; \mathbb{1}_{\omega'}; \tag{11}$$

$$d_1(t) = 0; \tag{12}$$

$$d_2(t) = \sqrt{\beta(t)}\; y^d(t)\; \mathbb{1}_{\omega'} \tag{13}$$

Then

$$J(u) = \frac{1}{2}\int_0^T \alpha(t)\; \|u(t)\|^2_{L^2(\omega)}\; dt \;+\; \frac{1}{2}\int_0^T \beta(t)\; \|y_u(t) - y^d(t)\|^2_{L^2(\omega')}\; dt.$$

Optimal control constructive characterisation

We have shown that the solution is of the form

$$\hat{u} = \nabla J^{\star}\left(-\mathcal{T}^{*}\hat{\varphi}^{T}\right), \tag{14}$$

where $\hat{\varphi}^{T}$ is the solution of minimisation problem (7).
For quadratic functional $J = \frac{1}{2}\|Cu - d\|^2_{\mathcal{X}}$ the formula (14) becomes

$$\hat{u} = \underbrace{-G\mathcal{B}^{*}e^{(t-T)\mathcal{A}^{*}}\hat{\varphi}^{T}}_{u_c} + \underbrace{GC^{*}d}_{\tilde{u}}, \tag{15}$$

where $G = (C^{*}C)^{-1}$, while $\hat{\varphi}^{T}$ is the minimiser of the problem (7).

We have to determine $\hat{\varphi}^{T}$.

Optimal control constructive characterisation

For $J = \frac{1}{2}\|Cu - d\|^2_{\mathcal{X}}$ we have

$$\hat{\varphi}^T = \arg\min_{\varphi^T \in \mathcal{H}} \left\{ F(\varphi^T) \right\} = \arg\min_{\varphi^T \in \mathcal{H}} \left\{ \langle M_T\ \varphi^T, \varphi^T \rangle_{\mathcal{H}} \right\}, \qquad (16)$$

where $M_t : \mathcal{H} \to \mathcal{H}$ is given by:

$$M_t\left(\varphi^T\right) = \int_0^t e^{(s-t)\mathcal{A}} \mathcal{B} \left\{ \left[(C^* C)^{-1} \left(\mathcal{B}^* e^{(\cdot - T)\mathcal{A}^*} \varphi^T \right) \right](s) \right\}\, ds. \qquad (17)$$

In addition

$$y(T) = -M_T \varphi^T + \tilde{y}(T).$$

Consequently, the original problem $(\mathcal{P})$ is equivalent to

$$(\mathcal{P}') \qquad \min_{\varphi^T \in \mathcal{H}} \left\{ \langle M_T\ \varphi^T, \varphi^T \rangle_{\mathcal{H}} : \ \| \underbrace{M_T\ \varphi^T - \tilde{y}(T)}_{-y(T)} + y^T \|_{\mathcal{H}} = \varepsilon \right\}. \qquad (18)$$

– a standard constrained optimisation problem.

Optimal control constructive characterisation

Introduce the Lagrange functional

$$\mathcal{L}\left(\varphi^T,\mu\right) = \langle M_T\ \varphi^T,\varphi^T\rangle_{\mathcal{H}} + \mu\left(\|M_T\ \varphi^T - \tilde{y}(T) + y^T\|^2_{\mathcal{H}} - \varepsilon^2\right).$$

The optimality condition gives

$$(M_T + M_T^*)\,\hat{\varphi}^T + 2\hat{\mu}M_T^*\left(M_T\ \hat{\varphi}^T - \tilde{y}(T) + y^T\right) = 0, \tag{19}$$

implying (M_T is symmetric)

$$\hat{\varphi}^T = \underbrace{\left[M_T\,(I + \hat{\mu}M_T)\right]^{-1}}_{R_{\hat{\mu}M_T}}\left[\hat{\mu}M_T\left(\tilde{y}(T) - y^T\right)\right].$$

The explicit expression

$$\hat{\varphi}^T = R_{\hat{\mu}M_T}\left[\hat{\mu}\left(\tilde{y}(T) - y^T\right)\right]$$

of the minimisator in terms of the given data and the unknown scalar $\hat{\mu}$.

Putting it in the constraint

$$\|M_T\ \hat{\varphi}^T - \tilde{y}(T) + y^T\|_{\mathcal{H}} = \varepsilon. \tag{20}$$

we get

$$\varepsilon = \|R_{\hat{\mu}M_T}\left(\tilde{y}(T) - y^T\right)\|_{\mathcal{H}}. \tag{21}$$

Geometrical interpretation

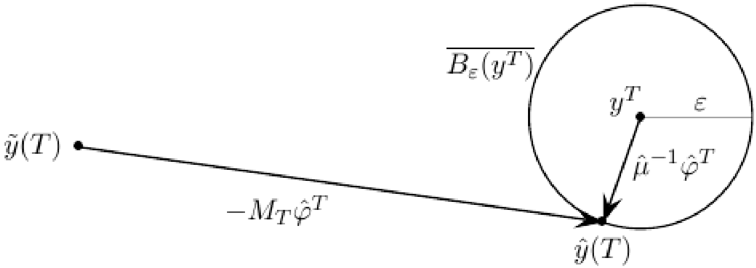


Figure: Geometrical interpretation of the optimal final state.

Optimal control $\hat{u}$ - expressed by optimal dual final-state $\hat{\varphi}^T$.

Optimal dual final-state $\hat{\varphi}^T$ – expressed by optimal Lagrange multiplier.

Optimal control constructive characterisation

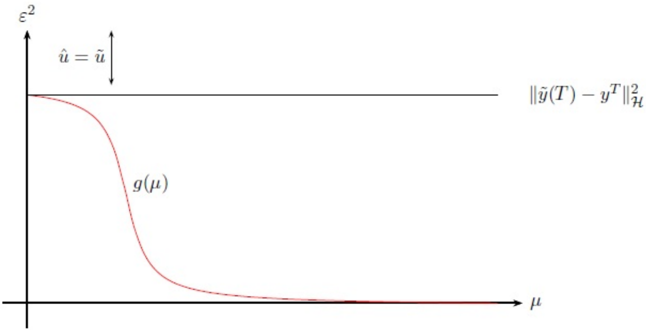
Let $g : \mathbf{R}^+ \to \mathbf{R}^+$ be given by

$$g(\mu) = \| R_{\mu M_T} \left(\tilde{y}(T) - y^T \right) \|_{\mathcal{H}}^2. \tag{22}$$

The problem is reduced to a scalar (nonlinear) equation

$$g(\hat{\mu}) = \varepsilon^2.$$

The equation is well defined for every $\varepsilon < \|\tilde{y}(T) - y^T\|$.



The constructive algorithm

- Find the real value $\hat{\mu}$ (optimal Lagrange multiplier) as the unique solution to

$$g(\hat{\mu}) = \varepsilon^2;$$

  for g given by (22).
- Find the vector $\hat{\varphi}^T \in \mathcal{H}$ (optimal dual final-state), as

$$\hat{\varphi}^T = R_{\hat{\mu}M_T}\left[\hat{\mu}\left(\tilde{y}(T) - y^T\right)\right],$$

  for $R_{\mu M_T} = (I + \mu M_T)^{-1}$ and M_T given by (17).
- Find the function $\hat{\varphi}(t)$ (optimal dual variable), given by

$$\hat{\varphi}(t) = e^{(t-T)\mathcal{A}^*}\hat{\varphi}^T.$$

- The optimal control is given by

$$\hat{u} = \underbrace{-G\mathcal{B}^*\hat{\varphi}}_{u_c} + \underbrace{GC^*d}_{\tilde{u}},$$

  where $G = (C^*C)^{-1}$.

Interpretation of u_c

The constrained component of the optimal control

$$u_c = -\underbrace{(C^*C)^{-1}}_{\text{Delicate part}} \mathcal{B}^*\hat{\varphi}\,.$$

We show

$$u_c = -\frac{1}{\alpha}\mathcal{B}^*\varphi$$

where φ is the solution to the system

$$\begin{cases} y' + \mathcal{A}y = -\frac{1}{\alpha}\mathcal{B}\mathcal{B}^*\varphi \\ y(0) = 0 \\ -\varphi' + \mathcal{A}\varphi = \beta y \\ \varphi(T) = \hat{\varphi}^T = \hat{\mu}(\hat{y}(T) - y^T). \end{cases} \tag{23}$$

This is the optimality system of the penalisation problem

$$\min_u \left\{ \frac{1}{2}\int_0^T \alpha(t)\|u(t)\|_{\mathcal{H}}^2 dt + \frac{1}{2}\int_0^T \beta(t)\|y_u(t)\|_{\mathcal{H}}^2 dt + \frac{\hat{\mu}}{2}\|y_u(T) - y^T\|_{\mathcal{H}}^2 \right\}.$$

Spectral decomposition

Denote:

$(\psi_n)_{n\in\mathbf{N}}$ – an orthonormal basis of H, consisting of eigenfunction of $\mathcal{A}$

$(\lambda_n)_{n\in\mathbf{N}}$ – a sequence of corresponding (nonnegative) eigenvalues λ_n, $\lim_n \lambda_n = +\infty$.

y_n – the n-th Fourier coefficient of $y \in H$.

The optimality system (23) can be rewritten as a 2$^{\mathrm{nd}}$ order ODE

$$-\varphi'' + \frac{\beta'}{\beta}\varphi' + (\mathcal{A}^2 - \frac{\beta'}{\beta}\mathcal{A} + \frac{\beta}{\alpha}\mathcal{B}_t\mathcal{B}_t^*)\varphi = 0\,.$$

If $\mathcal{B}_t\mathcal{B}_t^*$ is diagonalisable in the same basis of eigenfunctions of $\mathcal{A}$ the system can be solved component-wise.

Similarly, the operator M_T

$$M_T\left(\varphi^T\right) = \int_0^T e^{(s-T)\mathcal{A}}\mathcal{B}_s\left\{\left[(C^*C)^{-1}\left(\mathcal{B}^* e^{(\cdot-T)\mathcal{A}^*}\varphi^T\right)\right](s)\right\}\,ds,$$

can be presented by an infinite matrix with entries

$$(M_t)_{jk} = \int_0^T \left\langle\,(C^*C)^{-1}\left[\mathcal{B}^* e^{\lambda_j(\cdot-T)}\psi_j\right](s)\,,\;\mathcal{B}^* e^{\lambda_k(s-T)}\psi_k\right\rangle_{\mathcal{H}}\,ds. \qquad (24)$$

Truncation - required for practical implementation of the algo

Control cost- example

$$\min_{u \in L^2((0,L)\times(0,T))} \left\{ \int_0^T \alpha(t) \|u(t)\|^2_{L^2(0,L)} : \quad \|y(T) - y^T\|_{L^2(0,\pi)} \leq \varepsilon \right\}$$

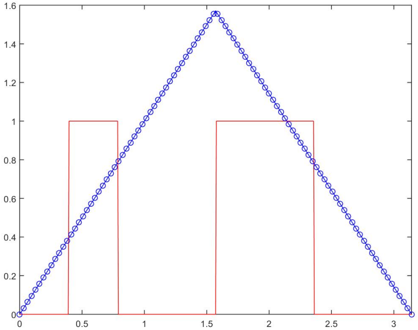
where:

- $\alpha = e^{5t}$, $L = \pi$, $T = 1$ and

$$\begin{cases} \partial_t y(x,t) - \partial_{xx} y(x,t) = u(x,t) \cdot \mathcal{I}_{\omega_c}(x) & x \in (0,L)\,,\ t \in [0,T] \\ y(0,t) = y(L,t) = 0 & t \in [0,T] \\ y(x,0) = 0 & x \in [0,L]\,, \end{cases}$$

  with $\omega_c = \left(\frac{L}{8}, \frac{L}{4}\right) \cup \left(\frac{L}{2}, \frac{3L}{4}\right)$;

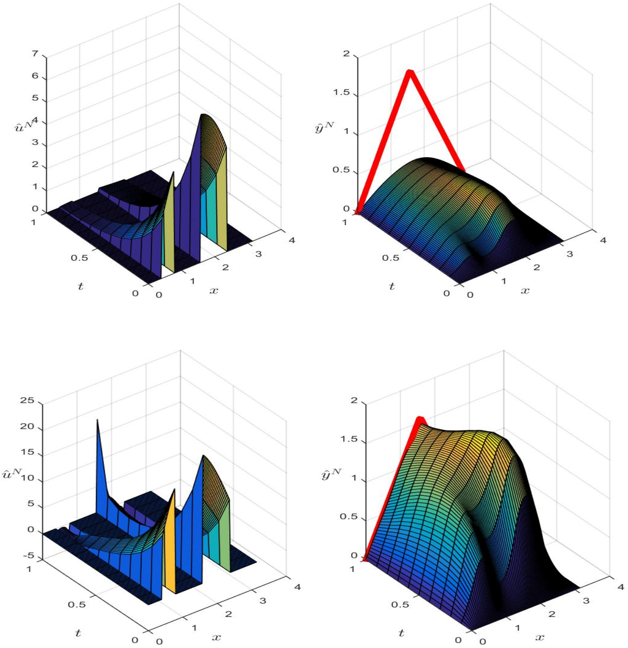
Exmple - Control cost



- final target (blue):

$$y^T(x) = \frac{L}{2} - \|x - \frac{L}{2}\|;$$

We used $\psi_n(x) = \sqrt{\frac{2}{L}} \cdot \sin(\frac{nx\pi}{L})$, $\lambda_n = n^2$ and $N = 230$.



$\varepsilon^2 = [1,\ 0.05\ ]$

Example - Trajectory regulation

$$\min_u \left\{ \alpha \int_0^T |u(t)|^2 \, dt + \beta \int_0^T \left| y(t) - y^d \right|^2 \, dt : \quad \|y(T) - y^T\|_{L^2(0,\pi)} \leq \varepsilon \right\},$$

where:

- equation: the same as before, but with $\omega_c = \Omega$;
- final target ($T = 1$):

$$y^T(x) = 3 \exp\left(-15 \left(x - \frac{3\pi}{4} \right)^2 \right);$$

- trajectory target: for $t_1 = \frac{2}{3}T$,

$$y^d(x,t) = 5 \exp\left(-15 \left(x - \frac{\pi}{4} \right)^2 \right) \mathcal{I}_{[0,t_1]}(t);$$

- $\beta = 1$;

We used $N = 25$.

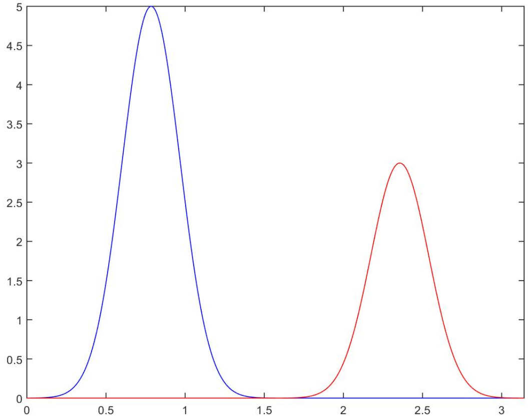


Figure: Red: y^T. Blue: y^d.

y^d is targeted just during $t \in [0, \frac{2}{3}]$.

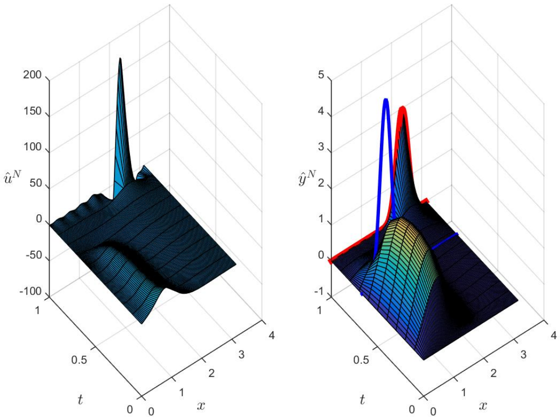


Figure: For $\alpha = 0.01$ and $\varepsilon^2 = 0.05$, the optimal control (Left) and the optimal state (Right). Red line: y^T. Blue line: y^d.

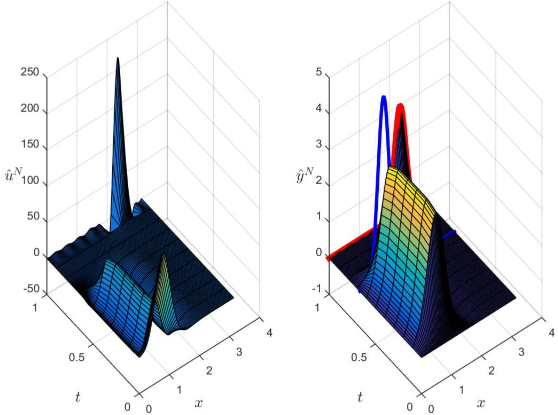


Figure: For $\alpha = 0.001$ and $\varepsilon^2 = 0.05$, the optimal control (Left) and the optimal state (Right). Red line: y^T. Blue line: y^d.

Conclusion

The new approach:

- exploring spectral representation of the solution by eigenfunctions of $\mathcal{A}$,
- an explicit expression of the optimal constrained control in terms of the given problem data.
- the numerical issues are reduced to finding the unique root of a scalar function.
- same formula applies independently of the dimension.
  No curse of dimensionality.

Price to pay:

- knowledge of eigenfunctions,
  If the problem has to be considered many times for different data, but the same operator, this can be done offline.

Thanks for your attention!