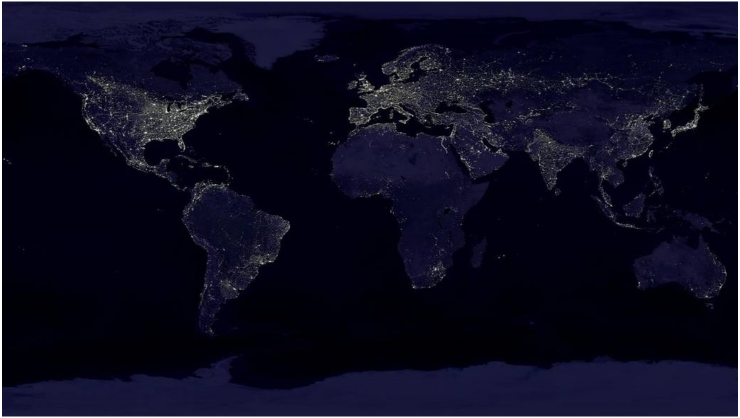
On Dissipativity in Analysis and Control of Large-scale Systems

Andrej Jokić

University of Zagreb, Croatia
Faculty of Mechanical Engineering and Naval Architecture

Joint work with **Ivica Nakić**

The biggest machine in the world



The biggest (most complex) machine in the world...
(Fair to say:) *...but it didn't feel like it ("the most complex") during design...*

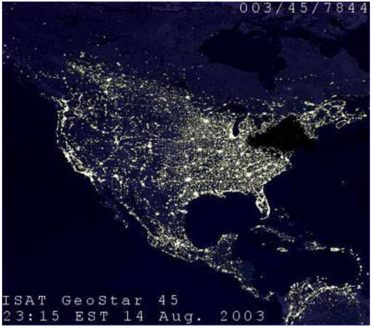
The biggest machine in the world

Robustness?

- synchronous machines, inertia (desirable physical properties)
- conservative engineering

... and fragility

- Blackout 9 November 1965, USA, Canada (tripping transmission line). Affected > 30 mil. people.
- Thailand 1978; Canada 1989; Brazil 1999 (97 mil. people); India 2001 (226 mil. people); USA & Canada 2003; Italy 2003; Germany & France & Italy & Spain 2006; China 2008; India 2012 (670 mil. people)



… things are changing…

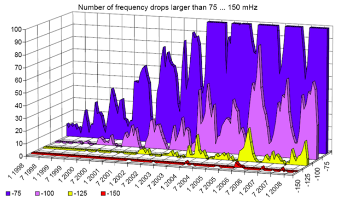
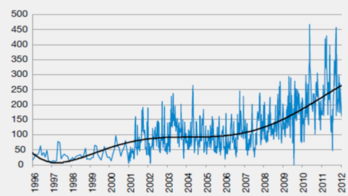
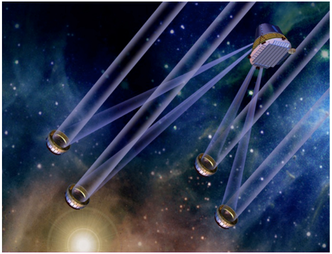


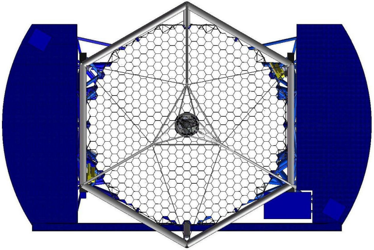
Figure 3.4: Evolution of number of 75, 100, 125, 150 mHz frequency drops



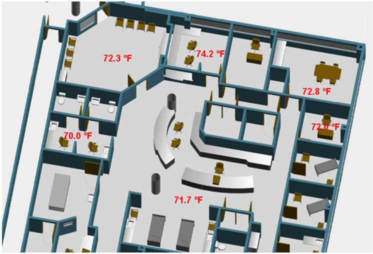
Number of frequency violations in Nordic grid (source: ENTSO-E)

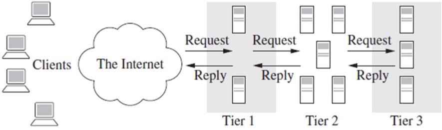
Distributed systems / dynamic networks

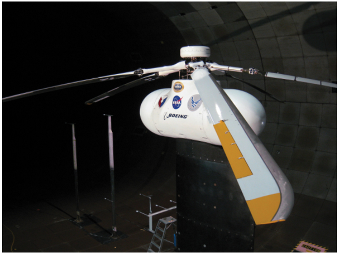


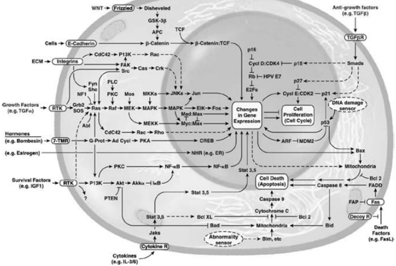


Distributed systems / dynamic networks









Outline

Outline

Introduction

Dissipativity, neutral supply functions, separation
- Dissipativity
- Interconnection neutral supply functions (2 systems)

Dynamical networks
- Acyclic networks
- Structured Lyapunov functions and robustness

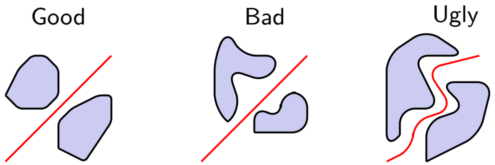
Beyond static supply rates (separation) & Synthesis

Architecture, Constraints that de-constrain (protocols) + Examples

Proofs... certificates of separation

Verifiable characterization of whether two sets are separated $\rightarrow$ in the heart of verity of problems.

- Hahn-Banach separation theorem
  - hyperplane separation in convex optimization (Lagrange multiplier)
- Separation of subspaces
  - Full block S-Procedure (multipliers)



Bigger picture: multipliers as protocols in architecture of dynamical networks

Full block S-Procedure in control systems has nice physical interpretation in terms of dissipativity theory.

Quadratic separation

Statement A (desired property)

For $G := \begin{pmatrix} I & G_2 \\ G_1 & I \end{pmatrix}$ inverse G^{-1} exists and $\|G^{-1}\| \leq c$

$$\Updownarrow$$

Statement B (Quadratic separation)

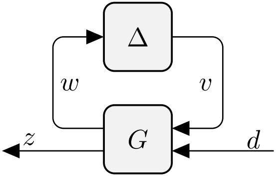
Exists Π such that $\begin{pmatrix} I \\ G_1 \end{pmatrix}^\top \Pi \begin{pmatrix} I \\ G_1 \end{pmatrix} \prec 0, \ \begin{pmatrix} G_2 \\ I \end{pmatrix}^\top \Pi \begin{pmatrix} G_2 \\ I \end{pmatrix} \succeq 0.$

Example

- $\begin{pmatrix} I & \frac{1}{s} I \\ A & I \end{pmatrix}^{-1}$ exists, is analytic with uniformly bounded norm in $\mathbb{C}^0 \cup \mathbb{C}^+$
- $\exists P \succ 0$ (note $\begin{pmatrix} \frac{1}{s} \\ I \end{pmatrix}^* \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{s} \\ I \end{pmatrix} \succeq 0$ on $\mathbb{C}^0 \cup \mathbb{C}^+$) :

  $\begin{pmatrix} I \\ A \end{pmatrix}^\top \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix} \prec 0$, i.e. $A^\top P + PA \prec 0$

Full block S-Procedure

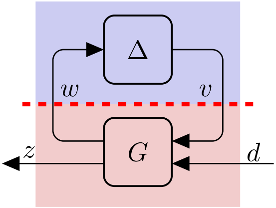


$$z = (\Delta \star G)d \;\; := \begin{cases} \begin{pmatrix} w \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} K & L \\ M & N \end{pmatrix}}_{G} \begin{pmatrix} v \\ d \end{pmatrix} \\ v = \Delta w, \quad \Delta \in \boldsymbol{\Delta} \end{cases}$$

Statement A (desired property)

$$\begin{pmatrix} d \\ z \end{pmatrix}^\top \begin{pmatrix} Q_p & S_p \\ S_p^\top & R_p \end{pmatrix} \begin{pmatrix} d \\ z \end{pmatrix} < 0 \quad \text{for all} \quad \begin{pmatrix} d \\ z \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} I \\ \Delta \star G \end{pmatrix}, \; \Delta \in \boldsymbol{\Delta}$$

Full block S-Procedure



$$z = (\Delta \star G)d \ := \begin{cases} \begin{pmatrix} w \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} K & L \\ M & N \end{pmatrix}}_{G} \begin{pmatrix} v \\ d \end{pmatrix} \\ v = \Delta w, \quad \Delta \in \boldsymbol{\Delta} \end{cases}$$

Statement B (multiplier-based condition - separation)

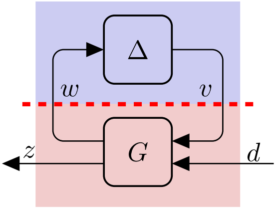
$$\begin{pmatrix} v \\ w \\ \hline 0 \\ 0 \end{pmatrix}^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^\top & R & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^\top & R_p \end{array}\right) \begin{pmatrix} v \\ w \\ \hline 0 \\ 0 \end{pmatrix} \geq 0 \quad \text{for all} \quad \Delta \in \boldsymbol{\Delta}$$

$$\begin{pmatrix} v \\ w \\ \hline d \\ z \end{pmatrix}^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^\top & R & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^\top & R_p \end{array}\right) \begin{pmatrix} v \\ w \\ \hline d \\ z \end{pmatrix} < 0$$

$$\Downarrow$$

Statement A (desired property)

Full block S-Procedure



$$z = (\Delta \star G)d \;\; := \;\; \begin{cases} \begin{pmatrix} w \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} K & L \\ M & N \end{pmatrix}}_{G} \begin{pmatrix} v \\ d \end{pmatrix} \\ v = \Delta w, \quad \Delta \in \boldsymbol{\Delta} \end{cases}$$

Statement B (multiplier-based condition - separation)

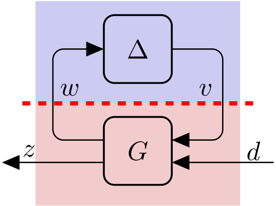
$$w^\top * / \quad \left(\begin{array}{c} \Delta \\ I \\ \hline 0 \\ 0 \end{array}\right)^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^\top & R & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^\top & R_p \end{array}\right) \left(\begin{array}{c} \Delta \\ I \\ \hline 0 \\ 0 \end{array}\right) \succeq 0 \text{ for all } \Delta \in \boldsymbol{\Delta} \quad / * w$$

$$\begin{pmatrix} v \\ d \end{pmatrix}^\top * / \quad \left(\begin{array}{cc} I & 0 \\ K & L \\ \hline 0 & I \\ M & N \end{array}\right)^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^\top & R & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^\top & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ K & L \\ \hline 0 & I \\ M & N \end{array}\right) \prec 0 \, / * \begin{pmatrix} v \\ d \end{pmatrix}$$

$$\Downarrow$$

Statement A (desired property)

Full block S-Procedure



Statement B (multiplier-based condition - separation)

$$\left(\begin{array}{c}\Delta\\ I\\ \hline 0\\ 0\end{array}\right)^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0\\ S^\top & R & 0 & 0\\ \hline 0 & 0 & Q_p & S_p\\ 0 & 0 & S_p^\top & R_p \end{array}\right)\left(\begin{array}{c}\Delta\\ I\\ \hline 0\\ 0\end{array}\right) \succeq 0 \quad \text{for all} \quad \Delta \in \boldsymbol{\Delta}$$

$$\left(\begin{array}{cc} I & 0\\ K & L\\ \hline 0 & I\\ M & N\end{array}\right)^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0\\ S^\top & R & 0 & 0\\ \hline 0 & 0 & Q_p & S_p\\ 0 & 0 & S_p^\top & R_p \end{array}\right)\left(\begin{array}{cc} I & 0\\ K & L\\ \hline 0 & I\\ M & N\end{array}\right) \prec 0$$

$$\Downarrow$$

Statement A (desired property)

Full block S-Procedure

Necessity: $\boldsymbol{\Delta}$ compact (Scherer 2001), extension (Jokić, Nakić 2017)

Statement B (multiplier-based condition - separation)

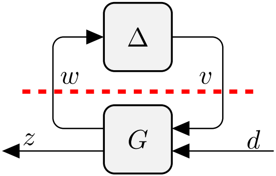
$$\begin{pmatrix} \Delta \\ I \\ \hline 0 \\ 0 \end{pmatrix}^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^\top & R & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^\top & R_p \end{array}\right) \begin{pmatrix} \Delta \\ I \\ \hline 0 \\ 0 \end{pmatrix} \succeq 0 \quad \text{for all} \quad \Delta \in \boldsymbol{\Delta}$$

$$\begin{pmatrix} I & 0 \\ K & L \\ \hline 0 & I \\ M & N \end{pmatrix}^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^\top & R & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^\top & R_p \end{array}\right) \begin{pmatrix} I & 0 \\ K & L \\ \hline 0 & I \\ M & N \end{pmatrix} \prec 0$$

$$\Updownarrow$$

Statement A (desired property)

$$\begin{pmatrix} d \\ z \end{pmatrix}^\top \begin{pmatrix} Q_p & S_p \\ S_p^\top & R_p \end{pmatrix} \begin{pmatrix} d \\ z \end{pmatrix} < 0 \quad \text{for all} \quad \begin{pmatrix} d \\ z \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} I \\ \Delta \star G \end{pmatrix}, \; \Delta \in \boldsymbol{\Delta}$$



$$z = (\Delta \star G) d \;\; := \;\; \begin{cases} \begin{pmatrix} w \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} K & L \\ M & N \end{pmatrix}}_{G} \begin{pmatrix} v \\ d \end{pmatrix} \\ v = \Delta w, \quad \Delta \in \boldsymbol{\Delta} \end{cases}$$

Statement B (multiplier-based condition)

$$\exists \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \; : \; \begin{cases} \begin{pmatrix} \Delta \\ I \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix} \succeq 0 \quad \text{for all} \quad \Delta \in \boldsymbol{\Delta} \\ \left(\begin{array}{cc} I & 0 \\ K & L \\ \hline 0 & I \\ M & N \end{array}\right)^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^\top & R & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^\top & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ K & L \\ \hline 0 & I \\ M & N \end{array}\right) \prec 0 \end{cases}$$

$$\Updownarrow$$

Statement A (desired property)

$$\begin{pmatrix} d \\ z \end{pmatrix}^\top \begin{pmatrix} Q_p & S_p \\ S_p^\top & R_p \end{pmatrix} \begin{pmatrix} d \\ z \end{pmatrix} < 0 \quad \text{for all} \quad \begin{pmatrix} d \\ z \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} I \\ \Delta \star G \end{pmatrix}, \; \Delta \in \boldsymbol{\Delta}$$

Example: robust stability

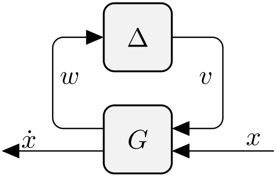
Uncertain dynamical system

$$\dot{x} = A(\delta)x, \ \delta \in \boldsymbol{\delta}.$$

Uniform exponential stability condition (single quadratic Lyapunov function)

$$\exists P \succ 0 \ : \ \frac{d}{dt}(x^\top P x) = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}^\top \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \leq 0, \ \dot{x} = A(\delta)x \text{ for all } \delta \in \boldsymbol{\delta}.$$

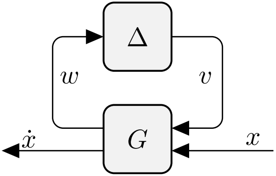
LFT representation of an uncertain system:



$$\begin{pmatrix} w \\ \dot{x} \end{pmatrix} = \underbrace{\begin{pmatrix} K & L \\ M & N \end{pmatrix}}_{G} \begin{pmatrix} v \\ x \end{pmatrix}$$

$$v = \Delta w, \quad \Delta \in \boldsymbol{\Delta}$$

Example: robust stability



$$\begin{pmatrix} w \\ \dot{x} \end{pmatrix} = \underbrace{\begin{pmatrix} K & L \\ M & N \end{pmatrix}}_{G} \begin{pmatrix} v \\ x \end{pmatrix}$$

$$v = \Delta w, \quad \Delta \in \boldsymbol{\Delta}$$

Statement A (desired property)

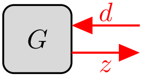
$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix}^\top \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \leq 0 \text{ for all } \Delta \in \boldsymbol{\Delta}.$$

Statement B (multiplier-based condition)

$$\exists \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} : \begin{cases} \begin{pmatrix} \Delta \\ I \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix} \succeq 0 \quad \text{for all} \quad \Delta \in \boldsymbol{\Delta} \\ \begin{pmatrix} I & 0 \\ K & L \\ \hline 0 & I \\ M & N \end{pmatrix}^\top \left(\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^\top & R & 0 & 0 \\ \hline 0 & 0 & 0 & P \\ 0 & 0 & P & 0 \end{array}\right) \begin{pmatrix} I & 0 \\ K & L \\ \hline 0 & I \\ M & N \end{pmatrix} \prec 0 \end{cases}$$

Outline

Dissipative dynamical systems (Jan Willems, 1972)



$$G : \begin{cases} \dot{x} = f(x, d) \\ z = g(x, d) \end{cases}$$

Dissipativity (global characterization)

Supply function: $s(d(t), z(t))$, storage function: $V(x(t))$

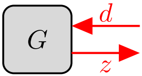
$$V(x(t_1)) + \int_{t_1}^{t_2} s(d(t), z(t))\mathrm{d}t \geq V(x(t_2))$$

Dissipativty (local characterization)

$$\partial_x V(x) f(x, d) \leq s(d(t), g(x, w))$$

$$\dot{V}(x) \leq s(d, z)$$

Dissipative dynamical systems (Jan Willems, 1972)



$$G : \begin{cases} \dot{x} = f(x, d) \\ z = g(x, d) \end{cases}$$

Strict dissipativity (global characterization)

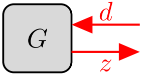
Supply function: $s(d(t), z(t))$, storage function: $V(x(t))$

$$V(x(t_1)) + \int_{t_1}^{t_2} s(d(t), z(t))\mathrm{d}t - \epsilon^2 \int_{t_1}^{t_2} \|d(t)\|^2 \mathrm{d}t \geq V(x(t_2))$$

Strict dissipativty (local characterization)

$$\dot{V}(x) \leq s(d, z) - \epsilon^2 \|d(t)\|^2$$

Dissipative dynamical systems



$$G: \begin{cases} \dot{x} &= Ax + Bd \\ z &= Cx + Dd \end{cases}$$

$$s(d,z) = -\begin{pmatrix} d \\ z \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} d \\ z \end{pmatrix}, \qquad V(x) = x^\top P x$$

Strict dissipativity

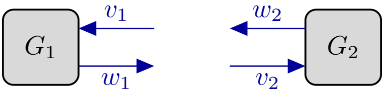
$$\underbrace{(Ax+Bd)^\top}_{\dot{x}^\top} Px + x^\top P \underbrace{(Ax+Bd)}_{\dot{x}} + (\star)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} d \\ Cx+Dd \end{pmatrix} < 0$$

Strict dissipativity

$$\left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right)^\top \left(\begin{array}{cc|cc} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right) \prec 0$$

Special cases: passivity, L_2 gain bound

Interconnection neutral supply functions



$$
\begin{aligned}
\dot{V}_1(x_1) &< s_1(v_1, w_1) \quad \text{for} \quad \mathrm{col}(x_1, v_1, w_1) \neq 0 \\
\dot{V}_2(x_2) &< s_2(v_2, w_2) \quad \text{for} \quad \mathrm{col}(x_2, v_2, w_2) \neq 0
\end{aligned}
$$

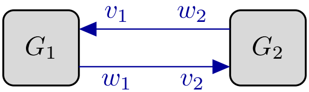
Interconnection neutral supply function

The interconnection is *neutral* with respect to supply rates s_1, s_2 if

$$s_1(v_1, w_1) + s_2(v_2, w_2) = 0,$$

for all v_1, w_1, v_2, w_2 such that $v_1 = w_2$, $v_2 = w_1$.

Interconnection neutral supply rates



Stability proof via neutral supply functions

$$
\begin{aligned}
&V_1(x_1) > 0, \quad \dot{V}_1(x_1) < s_1(v_1, w_1) \quad \text{for} \quad \mathrm{col}(x_1, v_1, w_1) \neq 0 \\
&V_2(x_2) > 0, \quad \dot{V}_2(x_2) < s_2(v_2, w_2) \quad \text{for} \quad \mathrm{col}(x_2, v_2, w_2) \neq 0
\end{aligned}
$$

$$
s_1(v_1, w_1) + s_2(v_2, w_2) = 0 \quad \text{for} \quad v_1 = w_2, v_2 = w_1
$$

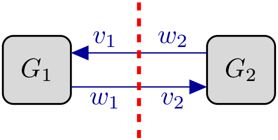
Willems, 1972 $\Downarrow$ Willems, 1972 $\Downarrow$ $\Uparrow$ Jokić, Nakić, 2016, 2017

Exists additive Lyapunov function

$V(x) = V_1(x_1) + V_2(x_2)$ is positive definite, $\dot{V}(x)$ is negative definite

Examples: passivity, small gain

Interconnection neutral supply rates



$$
(\star)^\top \left(\begin{array}{cc:cc} 0 & P_1 & 0 & 0 \\ P_1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A_1 & B_1 \\ \hdashline 0 & I \\ C_1 & D_1 \end{array}\right) \prec 0,
$$

$$
(\star)^\top \left(\begin{array}{cc:cc} 0 & P_2 & 0 & 0 \\ P_2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -R & -S^\top \\ 0 & 0 & -S & -Q \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A_2 & B_2 \\ \hdashline 0 & I \\ C_2 & D_2 \end{array}\right) \prec 0,
$$

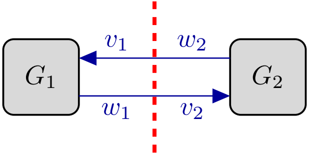
$P_1 \succ 0,\ P_2 \succ 0$

$\Updownarrow$ Jokić, Nakić, 2016, 2017

Additive Lyapunov function

$V(x) = x_1^\top P_1 x_1 + x_2^\top P_2 x_2$ is positive definite, $\dot{V}(x)$ is negative definite

Graph separation interpretation



(KYP lemma $\rightarrow$) Separation of graphs

$$\begin{pmatrix} I \\ G_1(s) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ G_1(s) \end{pmatrix} \prec 0 \quad \text{for all } s \in \mathbb{C}^0 \cup \mathbb{C}^+$$

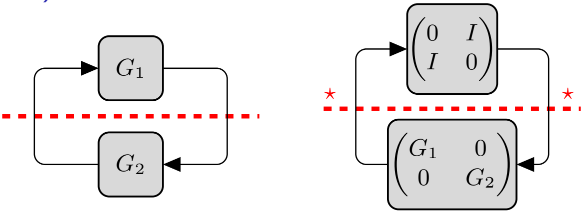
$$\begin{pmatrix} G_2(s) \\ I \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} G_2(s) \\ I \end{pmatrix} \succ 0 \quad \text{for all } s \in \mathbb{C}^0 \cup \mathbb{C}^+$$

$\Updownarrow$ Jokić, Nakić, 2016, 2017

Additive Lyapunov function

$V(x) = V_1(x_1) + V_2(x_2)$ is positive definite, $\dot{V}(x)$ is negative definite

Proof (sketch)



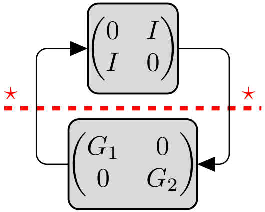
Given: There exists an additive Lyapunov function; $P = \mathrm{diag}(P_1, P_2)$.

$$H = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$, $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$, $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$

Problem: Search for a structured multiplier ("structured separation") $\star - \star$.

Proof (sketch)



Starting point: Full block S-procedure

$$\begin{pmatrix} H \\ I \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} H \\ I \end{pmatrix} \succeq 0$$

$$(\star)^\top \left(\begin{array}{cc:cc} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A & B \\ \hdashline 0 & I \\ C & D \end{array}\right) \prec 0.$$

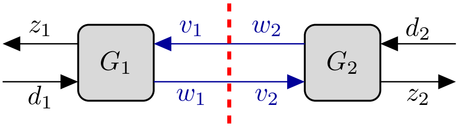
.

Proof: Full multiplier implies existence of a structured multiplier. Proof is constructive.

$$\begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} = \left(\begin{array}{cc:cc} Q_{11} & Q_{12} & S_{11} & S_{12} \\ Q_{12}^\top & Q_{22} & S_{21} & S_{22} \\ \hdashline S_{11}^\top & S_{21}^\top & R_{11} & R_{12} \\ S_{12}^\top & S_{22}^\top & R_{12}^\top & R_{22} \end{array}\right) \implies \exists \left(\begin{array}{cc:cc} \mathcal{Q} & 0 & \mathcal{S} & 0 \\ 0 & -\mathcal{R} & 0 & -\mathcal{S}^\top \\ \hdashline \mathcal{S}^\top & 0 & \mathcal{R} & 0 \\ 0 & -\mathcal{S} & 0 & -\mathcal{Q} \end{array}\right)$$

Assumption: Either C_1 and C_2 are full row rank or $D_1 = 0$, $D_2 = 0$.

Interconnection neutral supply rates for open systems



$$\dot V_1(x_1) < s_1(v_1, w_1) + s_1^{EX}(d_1, z_1) \quad \text{for} \quad \mathrm{col}(x_1, v_1, w_1) \neq 0$$
$$\dot V_2(x_2) < s_2(v_2, w_2) + s_2^{EX}(d_2, z_2) \quad \text{for} \quad \mathrm{col}(x_2, v_2, w_2) \neq 0$$

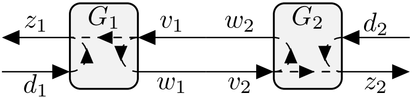
$$s_1(v_1, w_1) + s_2(v_2, w_2) = 0 \quad \text{for} \quad v_1 = w_2, v_2 = w_1$$

Willems, 1972 $\Downarrow$ $\Uparrow$ Jokić, Nakić, 2016, 2017

Dissipativity to external supply functions with additive storage function

$$\dot V(x) = \dot V_1(x_1) + \dot V_2(x_2) < s_1^{EX}(d_1, z_1) + s_2^{EX}(d_2, z_2)$$

Interconnection neutral supply rates for open systems



$$G_i : \begin{pmatrix} \dot{x}_i \\ w_i \\ z_i \end{pmatrix} = \begin{pmatrix} A_i & B_i & E_i \\ C_i & D_i & 0 \\ F_i & K_i & L_i \end{pmatrix} \begin{pmatrix} x_i \\ v_i \\ d_i \end{pmatrix}$$

Assumption: Either C_1 and C_2 are full row rank or $D_1 = 0$, $D_2 = 0$.

$$\dot{V}_1(x_1) < s_1(v_1, w_1) + s_1^{EX}(d_1, z_1) \quad \text{for} \quad \mathrm{col}(x_1, v_1, w_1) \neq 0$$

$$\dot{V}_2(x_2) < s_2(v_2, w_2) + s_2^{EX}(d_2, z_2) \quad \text{for} \quad \mathrm{col}(x_2, v_2, w_2) \neq 0$$

$$s_1(v_1, w_1) + s_2(v_2, w_2) = 0 \quad \text{for} \quad v_1 = w_2, v_2 = w_1$$

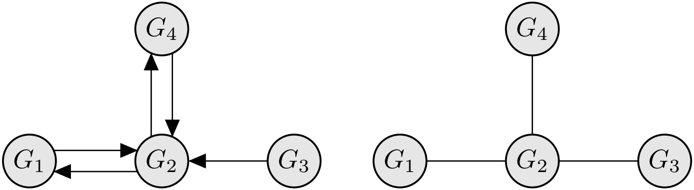
Willems, 1972 $\Downarrow$ $\Uparrow$ Jokić, Nakić, 2016, 2017

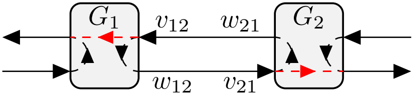
Dissipativity to external supply functions with additive storage function

$$\dot{V}(x) = \dot{V}_1(x_1) + \dot{V}_2(x_2) < s_1^{EX}(d_1, z_1) + s_2^{EX}(d_2, z_2)$$

Outline

Acyclic dynamical networks





$$G_i : \begin{pmatrix} \dot{x}_i \\ w_{iN_1} \\ w_{iN_2} \\ \vdots \\ w_{iN_{r(i)}} \end{pmatrix} = \begin{pmatrix} A_i & B_{iN_1} & B_{iN_2} & \dots & B_{iN_k} \\ C_{iN_1} & D_{iN_1} & 0 & \dots & 0 \\ C_{iN_2} & 0 & D_{iN_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ C_{iN_k} & 0 & 0 & \dots & D_{iN_{r(i)}} \end{pmatrix} \begin{pmatrix} x_i \\ v_{iN_1} \\ v_{iN_2} \\ \vdots \\ v_{iN_{r(i)}} \end{pmatrix}$$

Assumption: Either C_{ij} is full row rank or $D_{ij} = 0$.

Note: more general feed-through patterns - "implicitly" not acyclic networks

Additive Lyapunov functions and dissipativity

Statement 1: existence of an additive Lyapunov function

Dynamical network admits an additive quadratic Lyapunov function

$$V(x) = \underbrace{x_1^\top P_1 x_1}_{V_1(x_1)} + \dots \underbrace{x_L^\top P_L x_L}_{V_L(x_L)}$$

$$\Updownarrow$$

Statement 2: existence of interconnection neutral supply rates

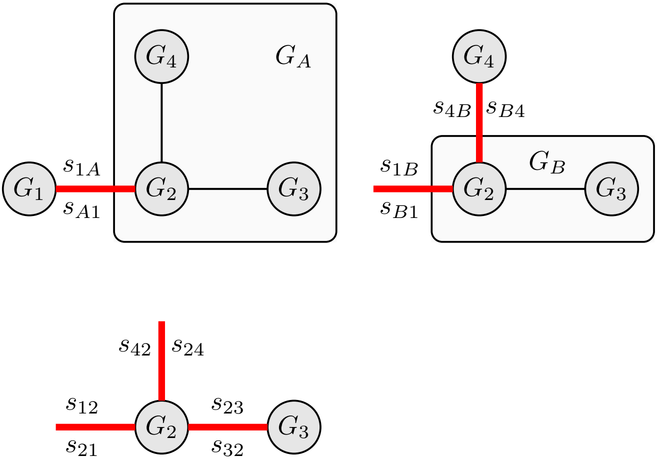
$$\dot{V}_i(x_i) < \sum_{j \in N_i} s_{ij}(v_{ij}, w_{ij})$$

along trajectories x_i, v_{ij}, w_{ij} satisfying

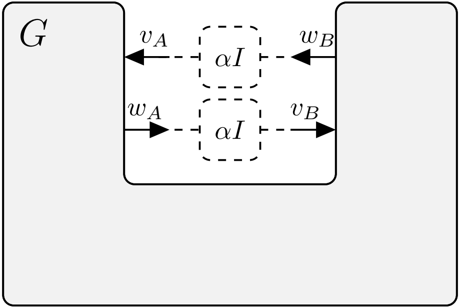
$$s_{ij}(v_{ij}, w_{ij}) + s_{ji}(v_{ji}, w_{ji}) = 0$$

for each (i, j) such that $(G_i, G_j) \in \hat{E}$;

Proof (illustration)



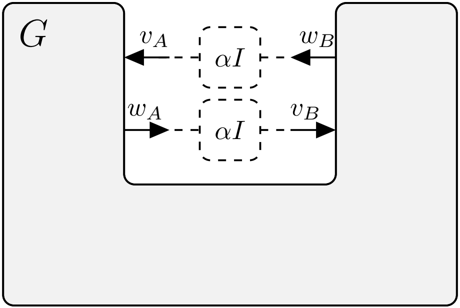
Robustness



Statement A

- G strictly dissipative w.r.t. $s(v, w) = s_A(v_A, w_A) + s_B(v_B, w_B)$ with storage $x^\top P x$
- $s_A(v_A, w_A) + s_B(v_B, w_B) = 0$ for all $v_A = w_B, v_B = w_A$
- $s_A(0, w_A) \leq 0$ for all $w_A \neq 0$
- $s_B(0, w_B) \leq 0$ for all $w_B \neq 0$
- $P \succ 0$

Robustness

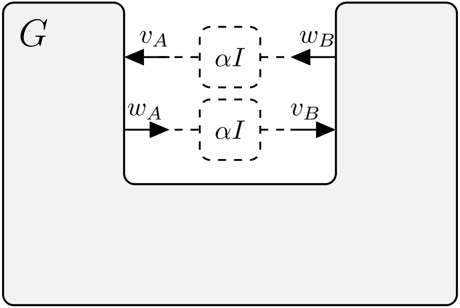


Statement B

- G is stable
- G remains stable if the following interconnection is made:
  $v_A = \alpha w_B$, $v_B = \alpha w_A$

Statement A $\implies$ Statement B

Robustness



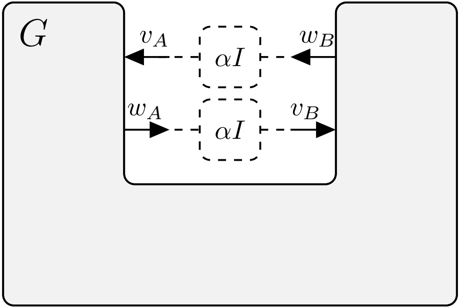
Proof. Direct application of the full block S-procedure

$$s(v,w) = -\begin{pmatrix} v_A \\ v_B \\ w_A \\ w_B \end{pmatrix}^\top \underbrace{\left(\begin{array}{cc|cc} Q & 0 & S & 0 \\ 0 & -R & 0 & -S^\top \\ \hline S^\top & 0 & R & 0 \\ 0 & -S & 0 & -Q \end{array}\right)}_{\Pi} \begin{pmatrix} v_A \\ v_B \\ w_A \\ w_B \end{pmatrix}, \quad R \succeq 0, \quad -Q \succeq 0$$

$s_A(v_A, w_A) + s_B(v_B, w_B) = 0$ for all $v_A = w_B, v_B = w_A$,

$s_A(0, w_A) \leq 0$ for all $w_A \neq 0$, $s_B(0, w_B) \leq 0$ for all $w_B \neq 0$

Robustness



Proof. Direct application of the full block S-procedure

$$\left(\begin{array}{cc} 0 & \alpha I \\ \alpha I & 0 \\ \hdashline I & 0 \\ 0 & I \end{array}\right)^{\top} \Pi \left(\begin{array}{cc} 0 & \alpha I \\ \alpha I & 0 \\ \hdashline I & 0 \\ 0 & I \end{array}\right) = \begin{pmatrix} (1-\alpha^2)R & \\ 0 & (1-\alpha^2)(-Q) \end{pmatrix} \succeq 0$$

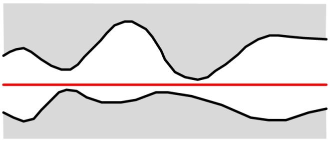
Robustness in dynamical networks

- Assumption: No algebraic loops $(D_{ij} = 0)$
- By construction: In acyclic networks exist interconnection neural supply rates which imply robustness w.r.t. loss of a link
- Networks with cycles: robustness w.r.t. to disconnection of a system (loss of all links which connect a system to the rest of the network)

Outline

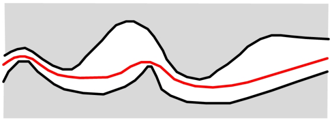
Beyond static supply rates (static separation)

For all $s \in \mathbb{C}^0 \cup \mathbb{C}^+$:



$$\begin{pmatrix} I \\ G_1(s) \end{pmatrix}^* \Pi \begin{pmatrix} I \\ G_1(s) \end{pmatrix} \succ 0$$

$$\begin{pmatrix} G_2(s) \\ I \end{pmatrix}^* \Pi \begin{pmatrix} G_2(s) \\ I \end{pmatrix} \prec 0$$

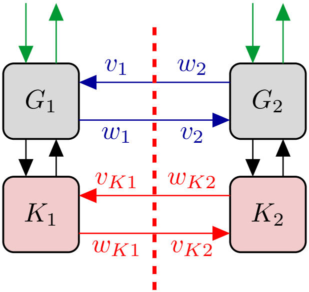


$$\begin{pmatrix} I \\ G_1(s) \end{pmatrix}^* \underbrace{\Psi(s)^* \Pi \Psi(s)}_{\Pi(s)} \begin{pmatrix} I \\ G_1(s) \end{pmatrix} \succ 0$$

$$\begin{pmatrix} G_2(s) \\ I \end{pmatrix}^* \underbrace{\Psi(s)^* \Pi \Psi(s)}_{\Pi(s)} \begin{pmatrix} G_2(s) \\ I \end{pmatrix} \prec 0$$

Note: IQCs, piecewise constant separation (e.g., in vertical layering), QDFs

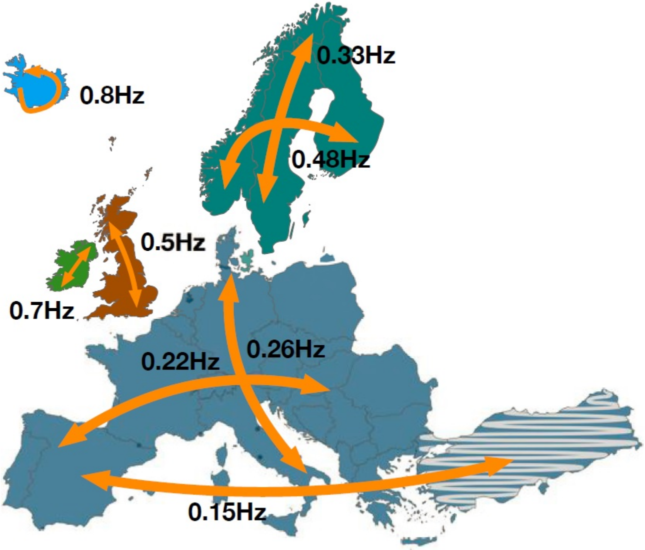
Synthesis (Distributed control)



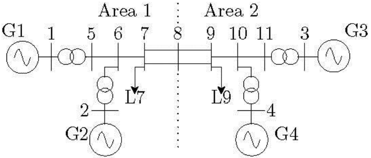
- Interactions in the control layer: more variables in interconnection neutral supplies
- Convexification in synthesis: equivalent to LPV control

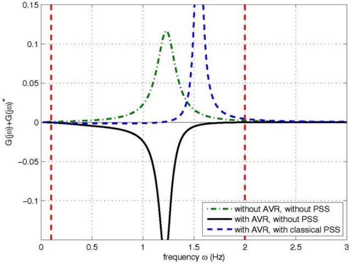
Outline

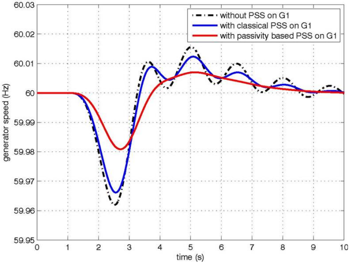
Motivating example: electrical power system



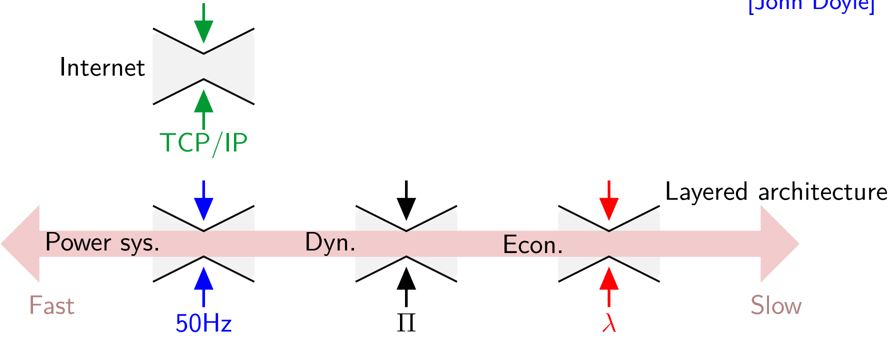
Motivating example: electrical power system

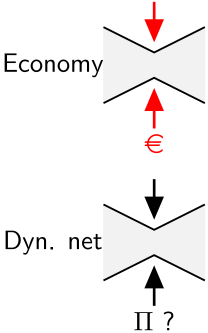






[John Doyle]



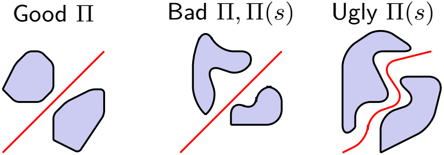


Domain specific knowledge!

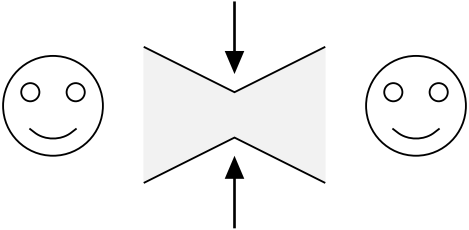
Hiding local complexity behind interface

No global model, confidentiality

Trade offs: roustness-efficiency-scalability



All happy families are alike;
each unhappy family is unhappy in its own way.



Acknowledgements

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