Distributed optimal control of parabolic equations by spectral decomposition

Martin Lazar
University of Dubrovnik

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Joint work with: C. Molinari,

The problem framework

The constrained minimisation problem

$$(\mathcal{P}) \qquad \min_u \left\{ J(u) : \quad y(T) \in \overline{B_\varepsilon\left(y^T\right)} \right\},$$

where:
– J is a given cost functional
– y^T is a given target
– y the solution of

$$(\mathcal{E}) \qquad \begin{cases} \frac{d}{dt} y(t) + \mathcal{A} y(t) = \mathcal{B}_t u(t) & \text{for } t \in (0,T) \\ y(0) = 0. \end{cases} \tag{1}$$

H1 The functional J is strictly convex, coercive and lower-semicontinuous.

H2 The unbounded linear operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is positive semidefinite, selfadjoint with dense domain $D(\mathcal{A})$ and compact resolvent.

H3 The operator $\mathcal{B}_t$ belongs to $\mathcal{L}(\mathcal{U}, \mathcal{H})$ for each time $t \in (0,T)$; moreover the pair (A, B_t) is approximately controllable in time T.

U, H - real Hilbert space

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M. Lazar, C. Molinari and J. Peypouquet: *Optimal control of parabolic equations by spectral decomposition*, Optimization, 23 pp, (2017)

The main example

Heat equation:

$$\begin{cases} \frac{d}{dt} y(t) - \Delta y(t) = \mathbb{1}_\omega u(t) & \text{in } \Omega \times (0, T) \\ y(t) = 0 & \text{on } \partial\Omega \times (0, T) \\ y(0) = 0 & \text{in } \Omega. \end{cases} \tag{2}$$

The system (2) is not exactly controllable.
For any open subset ω of positive measure system (2) is approximately controllable in any time $T > 0$.
The goal: among all the eligible controls to detect one minimising given cost functional.

Existence of the solution

Unconstrained problem:

$$\tilde{u} = \arg \min_{u \in L^2_{T,\mathcal{U}}} J(u). \tag{3}$$

It admits the unique solution $\tilde{u}$ (due to assumptions on J).

Theorem

The constrained problem $(\mathcal{P})$ admits a unique solution that we denote by $\hat{u}$.

If $\|\tilde{y}(T) - y^T\| \leq \varepsilon$, then the optimal control coincides with the solution of the unconstrained problem, i.e. $\hat{u} = \tilde{u}$.

Otherwise, the optimal final state verifies $\|\hat{y}(T) - y^T\|_{\mathcal{H}} = \varepsilon$ (i.e.: $\hat{y}(T)$ lies on $\partial B_\varepsilon(y^T)$).

In the sequel we suppose that $\varepsilon < \|\tilde{y}(T) - y^T\|$.

Characterisation of the solution by the dual problem

We introduce the Fenchel conjugate $J^\star$ of the functional J:

$$J^\star(u^\star) = \sup_{u \in L^2_{T,\mathcal{U}}} \{ \langle u^\star, u \rangle_{T,\mathcal{U}} - J(u) \} \quad \text{for } u^\star \in L^2_{T,\mathcal{U}}.$$

Theorem [Generalized HUM]

Let $\bar{y} \in \mathcal{H}$ be a reachable state.
Then

$$\bar{u} \in \arg\min_{u \in \mathcal{U}} \{ J(u) : \ \mathcal{T} u = \bar{y} \}. \tag{4}$$

is of the form $\bar{u} = \nabla J^\star\left(-\mathcal{T}^* \bar{\varphi}^T\right)$, where

$$\bar{\varphi}^T \in \arg\min_{\varphi^T \in \mathcal{H}} \left\{ J^\star(-\mathcal{T}^* \varphi^T) + \langle \bar{y}, \varphi^T \rangle_{\mathcal{H}} \right\}. \tag{5}$$

$\mathcal{T} : L^2_{T,\mathcal{U}} \to \mathcal{H}$ is the operator that takes the distributed control and gives the corresponding final state

$$\mathcal{T} u = y(T).$$

$$\mathcal{T}^* \varphi^T = \mathcal{B}^* \varphi$$

Characterisation of the solution by the dual problem

It is enough to restrict minimisation problem $(\mathcal{P})$ to controls of form $u = \nabla J^\star \left(-\mathcal{T}^* \varphi^T\right)$.
For such u

$$J(u) = F(\varphi^T),$$

where

$$F(\varphi^T) = -\left[\langle \nabla J^\star \left(-\mathcal{T}^* \varphi^T\right), \mathcal{T}^* \varphi^T \rangle_{L^2_{T,\mathcal{U}}} + J^\star \left(-\mathcal{T}^* \varphi^T\right)\right]. \quad (6)$$

Theorem

The solution of problem $(\mathcal{P})$ is

$$\hat{u} = \nabla J^\star \left(-\mathcal{T}^* \hat{\varphi}^T\right),$$

where $\hat{\varphi}^T$ is a solution of

$$\min_{\varphi^T \in \mathcal{H}} \left\{ F(\varphi^T) : \quad \|y(T) - y^T\|_{\mathcal{H}} = \varepsilon. \right\}. \quad (7)$$

Quadratic cost-functional

$$J(u) = \frac{1}{2}\|Cu - d\|^2_{\mathcal{X}}, \tag{8}$$

C – a linear bounded operator from $L^2_{T,\mathcal{U}}$ to a generic Hilbert space $\mathcal{X}$
We suppose that C is *uniformly elliptic*:

$$\|Cu\|_{\mathcal{X}} \geq \gamma \|u\|_{L^2_{T,\mathcal{U}}}. \tag{9}$$

It implies that it exists $(C^*C)^{-1}$.

EXAMPLE

Set $C = (C_1, C_2)$ and $d = (d_1, d_2)$:

$$(C_1 u)(t) = \sqrt{\alpha(t)}\; u(t)\; \mathbb{1}_{\omega}; \tag{10}$$

$$(C_2 u)(t) = \sqrt{\beta(t)}\; y_u(t)\; \mathbb{1}_{\omega'}; \tag{11}$$

$$d_1(t) = 0; \tag{12}$$

$$d_2(t) = \sqrt{\beta(t)}\; y^d(t)\; \mathbb{1}_{\omega'} \tag{13}$$

Then

$$J(u) = \frac{1}{2}\int_0^T \alpha(t)\; \|u(t)\|^2_{L^2(\omega)}\; dt \;+\; \frac{1}{2}\int_0^T \beta(t)\; \|y_u(t) - y^d(t)\|^2_{L^2(\omega')}\; dt.$$

Optimal control constructive characterisation

We have shown that the solution is of the form

$$\hat{u} = \nabla J^{\star}\left(-\mathcal{T}^{*}\hat{\varphi}^{T}\right), \tag{14}$$

where $\hat{\varphi}^{T}$ is the solution of minimisation problem (7).
For quadratic functional $J = \frac{1}{2}\|Cu - d\|_{\mathcal{X}}^{2}$ the formula (14) becomes

$$\hat{u} = -G\mathcal{B}^{*}e^{(t-T)\mathcal{A}^{*}}\hat{\varphi}^{T} + GC^{*}d, \tag{15}$$

where $G = (C^{*}C)^{-1}$, while $\hat{\varphi^{T}}$ is the minimiser of the problem (7).

We have to determine $\hat{\varphi}^{T}$.

Optimal control constructive characterisation

For $J = \frac{1}{2}\|Cu - d\|^2_{\mathcal{X}}$ we have

$$\hat{\varphi}^T = \arg\min_{\varphi^T \in \mathcal{H}} \left\{ F(\varphi^T) \right\} = \arg\min_{\varphi^T \in \mathcal{H}} \left\{ \langle M_T\, \varphi^T, \varphi^T \rangle_{\mathcal{H}} \right\}, \tag{16}$$

where $M_t : \mathcal{H} \to \mathcal{H}$ is given by:

$$M_t\left(\varphi^T\right) = \int_0^t e^{(s-t)\mathcal{A}} \mathcal{B} \left\{ \left[(C^*C)^{-1} \left(\mathcal{B}^* e^{(\cdot - T)\mathcal{A}^*} \varphi^T \right) \right](s) \right\}\, ds. \tag{17}$$

In addition

$$y(T) = -M_T \varphi^T + \tilde{y}(T).$$

Consequently, the original problem $(\mathcal{P})$ is equivalent to

$$(\mathcal{P}') \qquad \min_{\varphi^T \in \mathcal{H}} \left\{ \langle M_T\, \varphi^T, \varphi^T \rangle_{\mathcal{H}} : \quad \| \underbrace{M_T\, \varphi^T - \tilde{y}(T)}_{y(T)} + y^T \|_{\mathcal{H}} = \varepsilon \right\}. \tag{18}$$

– a standard constrained optimisation problem.

Optimal control constructive characterisation

Introduce the Lagrange functional

$$\mathcal{L}\left(\varphi^T, \mu\right) = \langle M_T\ \varphi^T, \varphi^T \rangle_{\mathcal{H}} + \mu \left(\|M_T\ \varphi^T - \tilde{y}(T) + y^T\|^2_{\mathcal{H}} - \varepsilon^2 \right).$$

The optimality condition gives

$$(M_T + M_T^*)\, \hat{\varphi}^T + 2\hat{\mu} M_T^* \left(M_T\ \hat{\varphi}^T - \tilde{y}(T) + y^T \right) = 0, \qquad (19)$$

implying

$$\hat{\varphi}^T = \underbrace{\left[M_T \left(I + \hat{\mu} M_T \right) \right]^{-1}}_{R_{\hat{\mu} M_T}} \left[\hat{\mu} M_T \left(\tilde{y}(T) - y^T \right) \right].$$

The explicit expression

$$\hat{\varphi}^T = R_{\hat{\mu} M_T} \left[\hat{\mu} \left(\tilde{y}(T) - y^T \right) \right]$$

of the minimisator in terms of the given data and the unknown scalar $\hat{\mu}$.

Putting it in the constraint

$$\|M_T\ \hat{\varphi}^T - \tilde{y}(T) + y^T\|_{\mathcal{H}} = \varepsilon. \qquad (20)$$

we get

$$\varepsilon = \|R_{\hat{\mu} M_T} \left(\tilde{y}(T) - y^T \right)\|_{\mathcal{H}}. \qquad (21)$$

Optimal control constructive characterisation

Let $g : \mathbf{R}^+ \to \mathbf{R}^+$ be given by

$$g(\mu) = \| R_{\mu M_T} \left(\tilde{y}(T) - y^T \right) \|_{\mathcal{H}}^2. \tag{22}$$

The problem is reduced to a scalar (nonlinear) equation

$$g(\hat{\mu}) = \varepsilon^2.$$

The equation is well defined for every $\varepsilon < \|\tilde{y}(T) - y^T\|$.

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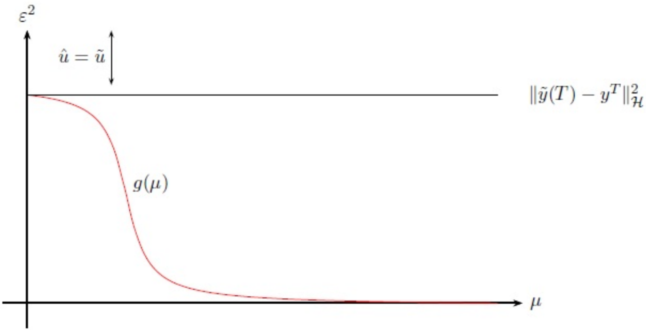

The constructive algorithm

- Find the real value $\hat{\mu}$ (optimal Lagrange multiplier) as the unique solution to
$$g(\hat{\mu}) = \varepsilon^2;$$
for g given by (22).
- Find the vector $\hat{\varphi}^T \in \mathcal{H}$ (optimal dual final-state), as
$$\hat{\varphi}^T = R_{\hat{\mu} M_T} \left[\hat{\mu} \left(\tilde{y}(T) - y^T \right) \right],$$
for $R_{\mu M_T} = (I + \mu M_T)^{-1}$ and M_T given by (17).
- Find the function $\hat{\varphi}(t)$ (optimal dual variable), given by
$$\hat{\varphi}(t) = e^{(t-T)\mathcal{A}^*} \hat{\varphi}^T.$$
- The optimal control is given by
$$\hat{u} = -G\mathcal{B}^* \hat{\varphi} + GC^* d,$$
where $G = (C^* C)^{-1}$.

Spectral decomposition

Denote:
$(\psi_n)_{n\in\mathbf{N}}$ – an orthonormal basis of H, consisting of eigenfunction of $\mathcal{A}$
$(\lambda_n)_{n\in\mathbf{N}}$ – a sequence of corresponding (nonnegative) eigenvalues λ_n,
$\lim_n \lambda_n = +\infty$.
y_n – the n-th Fourier coefficient of $y \in H$.

The operator M_T determined by

$$M_T\left(\varphi^T\right) = \int_0^T e^{(s-T)\mathcal{A}}\mathcal{B}_s\left\{\left[(C^*C)^{-1}\left(\mathcal{B}^* e^{(\cdot-T)\mathcal{A}^*}\varphi^T\right)\right](s)\right\}\, ds,$$

can be presented by an infinite matrix with entries

$$(M_t)_{jk} = \int_0^T \left\langle (C^*C)^{-1}\left[\mathcal{B}^* e^{\lambda_j(\cdot-T)}\psi_j\right](s),\ \mathcal{B}^* e^{\lambda_k(s-T)}\psi_k\right\rangle_{\mathcal{H}} ds, \qquad (23)$$

. Truncation - required for practical implementation of the algo

Truncation and error estimates

For $\varphi \in \mathcal{H}$ define its truncation

$$\varphi^N = \sum_{j=1}^{N} \varphi_j \ \psi_j ,$$

and similarly $M_T^N : \mathbf{R}^N \to \mathbf{R}^N$

$$\left(M_T^N\right)_{ij} = M_T \psi_i \cdot \psi_j , \quad , i, j = 1..N .$$

Define also

$$R^N(\mu) = \left(I + \mu M_T^N\right)^{-1}$$

and

$$g^N(\mu) = \| R^N [\tilde{y}(T) - y^T]^N \|^2_{\mathbf{R}^n} . \tag{24}$$

g^N preserves the "good" properties of g.

Truncation and error estimates

The equation

$$g^N(\mu) = \varepsilon^2$$

is well posed for every $\varepsilon < \|[\tilde{y}(T) - y^T]^N\|^2_{\mathbf{R}^n}$.

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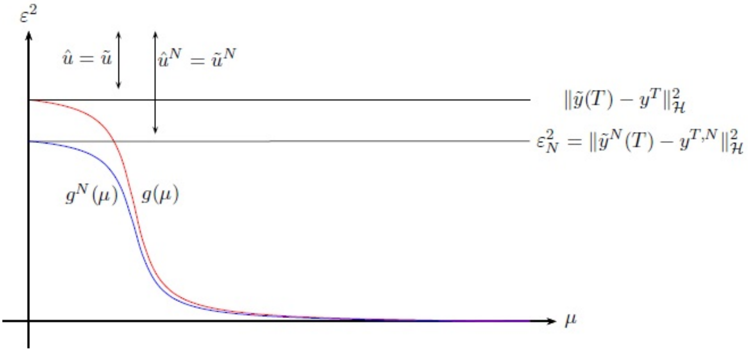


The algo -truncated version

- Approximation $\hat{\mu}^N$ of the optimal Lagrange multiplier as the unique solution to
$$g^N(\hat{\mu}) = \varepsilon^2;$$
for g^N given by (24).
- the approximation of the optimal dual final state
$$\left(\hat{\varphi}^T\right)^{\star,N} = \hat{\mu}^N R^N(\hat{\mu}^N)\left[\tilde{y}(T) - y^T\right]^N;$$

Important

$$\left(\hat{\varphi}^T\right)^{\star,N} \neq \left(\hat{\varphi}^T\right)^N, \quad \hat{u}^{\star,N} \neq \hat{u}^N$$

(nonlinear effects!).

- the approximation of the optimal dual variable
$$\hat{\varphi}^{\star,N}(t) = e^{(t-T)\mathcal{A}^*}\hat{\varphi}^T.$$
- the approximation of the optimal control
$$\hat{u}^{\star,N} = -G\mathcal{B}^*\hat{\varphi}^{\star,N} + GC^*d^N,$$
where $G = (C^*C)^{-1}$.

Much *boring* work to get what is expected
$$\hat{u}^{\star,N} \approx \hat{u}^N$$

Estimates

The problem is reduced to determining

$$R^N(\mu) = \left(I + \mu M_T^N\right)^{-1},$$

i.e. to the entries of the matrix M_T^N:

$$(M_T)_{jk} = \int_0^T \left\langle \underbrace{(C^*C)^{-1}\left[\mathcal{B}^* e^{\lambda_j(\cdot - T)}\psi_j\right](s)}_{\text{delicate part}},\ \mathcal{B}^* e^{\lambda_k(s-T)}\psi_k \right\rangle_{\mathcal{H}} ds.$$

Control cost

The problem is easy if there is no trajectory regulation:

$$J(u) = \frac{1}{2} \sum_k \int_0^T a_k(t) \left[u(t) - u^d(t) \right]_k^2 \, dt \tag{25}$$

in which case

$$\left[C_1 u \right](t) = \sum_k \sqrt{\alpha_k(t)} \, \left[u(t) \right]_k \, \psi_k;$$

Clearly, $C_1^* = C_1$ and

$$\left[C_1^* C_1 u \right](t) = \sum_k \alpha_k(t) \, \left[u(t) \right]_k \, \psi_k. \tag{26}$$

Control cost- example

Put $\alpha_k = e^{5t}$.

$$(\mathcal{P}) \qquad \min_{u\in L^2((0,L)\times(0,T))} \left\{ \int_0^T e^{5t} \|u(t)\|^2_{L^2(0,L)} : \quad \|y(T) - y^T\|_{L^2(0,\pi)} \leq \varepsilon \right\},$$

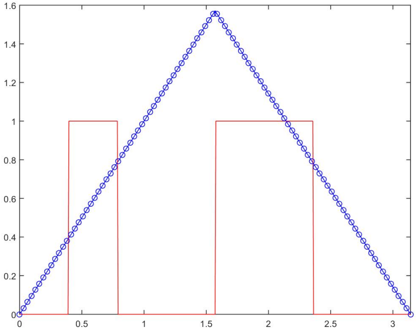
where:

- equation: $L = \pi$, $T = 1$ and

$$\begin{cases} \partial_t y(x,t) - \partial_{xx} y(x,t) = u(x,t) \cdot \mathcal{I}_{\omega_c}(x) & x \in (0,L)\,,\ t \in [0,T] \\ y(0,t) = y(L,t) = 0 & t \in [0,T] \\ y(x,0) = 0 & x \in [0,L]\,, \end{cases}$$

with $\omega_c = \left(\frac{L}{8}, \frac{L}{4}\right) \cup \left(\frac{L}{2}, \frac{3L}{4}\right)$;

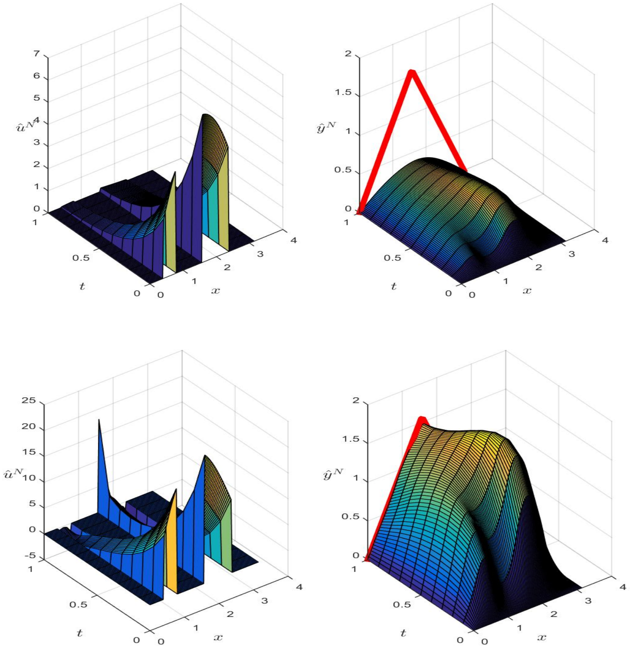
Exmple - Control cost



- final target (blue):

$$y^T(x) = \frac{L}{2} - \|x - \frac{L}{2}\|;$$

We used $\psi_n(x) = \sqrt{\frac{2}{L}} \cdot \sin(\frac{nx\pi}{L})$, $\lambda_n = n^2$ and $N = 230$.



$$\varepsilon^2 = [\ 0.5,\ 0.001\ ] \cdot \varepsilon_N^2.$$

Trajectory regulation

$$\min_u \left\{ \int_0^T \alpha(t) \left| u\left(t\right) \right|^2 \; dt + \int_0^T \beta(t) \left| y\left(t\right) - y^d \right|^2 \; dt : \quad \| y(T) - y^T \|_{L^2(0,\pi)} \leq \varepsilon \right\},$$

$C = (C_1, C_2)$ where

$$\left[C_1 u\right](t) = \sqrt{\alpha(t)} u(t); \tag{27}$$

$$\left[C_2 u\right](t) = \sqrt{\beta(t)} y_u(t); . \tag{28}$$

We know $C^* \left(a_1, a_2\right) = C_1^* a_1 + C_2^* a_2$ and we have to find

$$G = \left(C_1^* C_1 + C_2^* C_2\right)^{-1}. \tag{29}$$

More precisely

$$G \left[\mathcal{B}^* e^{\lambda_j (\cdot - T)} \psi_j \right].$$

Trajectory regulation

$$(C_1^* C_1 + C_2^* C_2)\, u = \alpha(t) u(t) + \int_t^T \int_0^s \beta(s) B^* e^{(t-s)A^*} \; e^{(q-s)A} B u(q) dq ds$$

What is its inverse?
Try the solution of

$$(C_1^* C_1 + C_2^* C_2)\, u(t) = \mathcal{B}^* e^{(t-T)A} \psi_j$$

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$$(C_1^* C_1 + C_2^* C_2)\, u(t) = \mathcal{B}^* e^{(t-T)A} \psi_j$$

in the form

$$u = \mathcal{B}^* \psi_j \, v_j(t),$$

for some unknown function v_j.
It is sufficient that

$$\alpha(t) v_j(t) \psi_j + + \int_t^T \int_0^s \beta(s) e^{(t-s)A^*} \; e^{(q-s)A} B \mathcal{B}^* \psi_j v_j(q) = e^{(t-T)A} \psi_j$$

Manipulation and double derivation with respect to time

$$\left(\ddot{v}_j(t) + \left[2\frac{\dot{\alpha}(t)}{\alpha(t)} - \frac{\dot{\beta}(t)}{\beta(t)} \right] \dot{v}_j(t) + \left[-\lambda_j^2 + \frac{\ddot{\alpha}(t)}{\alpha(t)} - \frac{\dot{\alpha}(t)\dot{\beta}(t)}{\alpha(t)\beta(t)} + \lambda \frac{\dot{\beta}(t)}{\beta(t)} \right] \; v_j(t) \right) \psi_j$$
$$- \frac{\beta(t)}{\alpha(t)} v_j(t) e^{-t\lambda_j} e^{tA} \mathcal{B}\mathcal{B}^* \psi_j = 0.$$

Trajectory regulation

Assume $\mathcal{B}\mathcal{B}^*$ diagonalisable (strong assumption).

$$\ddot{v}_j(t) + \left[2\frac{\dot{\alpha}(t)}{\alpha(t)} - \frac{\dot{\beta}(t)}{\beta(t)}\right]\dot{v}_j(t) + \left[-\lambda_j^2 + \frac{\ddot{\alpha}(t)}{\alpha(t)} - \frac{\dot{\alpha}(t)\dot{\beta}(t)}{\alpha(t)\beta(t)} + \lambda\frac{\dot{\beta}(t)}{\beta(t)} - \mu_j\frac{\beta(t)}{\alpha(t)}\right]v_j = 0.$$

Second order ODE for v_j.

We solve the problem for α, β:

- constant functions
- exponential functions
- characteristic funstions of a time interval

We obtain the missing puzzle of the algo - entries of the matrix M_T:

$$(M_T)_{jk} = \int_0^t \Big\langle \underbrace{G\left[\mathcal{B}^* e^{\lambda_j(\cdot - T)}\psi_j\right](s)}_{\mathcal{B}^*\psi_j v_j(t)}, \mathcal{B}^* e^{\lambda_k(s-T)}\psi_k \Big\rangle_{\mathcal{H}} \, ds.$$

Example - Trajectory regulation

$$\min_u \left\{ \alpha \int_0^T |u(t)|^2 \ dt + \beta \int_0^T \left| y(t) - y^d \right|^2 \ dt : \quad \|y(T) - y^T\|_{L^2(0,\pi)} \le \varepsilon \right\},$$

where:

- equation: the same as before, but with $\omega_c = \Omega$;
- final target ($T = 1$):

$$y^T(x) = 3 \exp\left(-15 \left(x - \frac{3\pi}{4} \right)^2 \right);$$

- trajectory target: for $t_1 = \frac{2}{3}T$,

$$y^d(x,t) = 5 \exp\left(-15 \left(x - \frac{\pi}{4} \right)^2 \right) \mathcal{I}_{[0,t_1]}(t);$$

- $\beta = 1$;

We used $N = 25$.

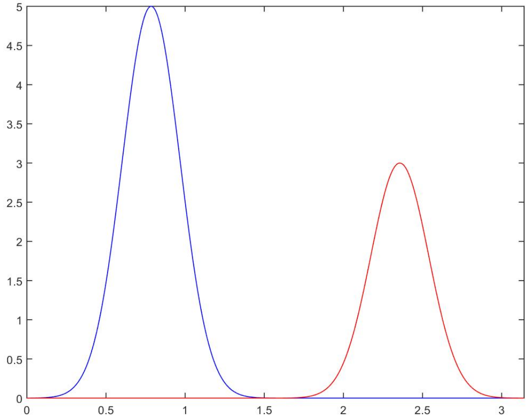


Figure: Red: y^T. Blue: y^d.

y^d is targeted just during $t \in [0, \frac{2}{3}]$.

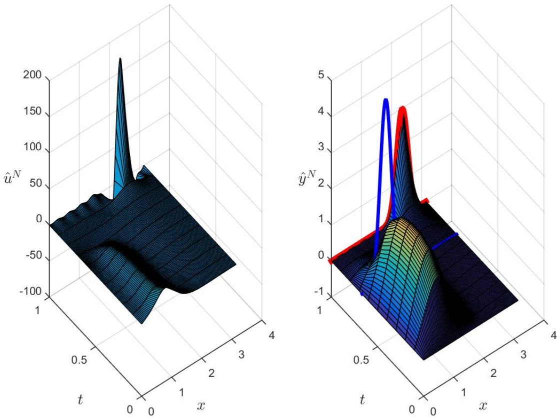
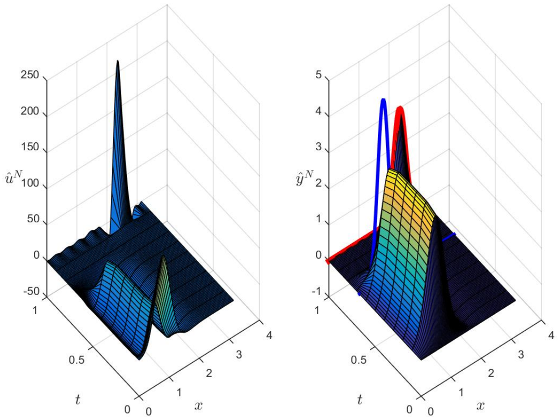


Figure: For $\alpha = 0.01$ and $\varepsilon^2 = 0.001\ g(0)$, the optimal control (Left) and the optimal state (Right). Red line: y^T. Blue line: y^d.



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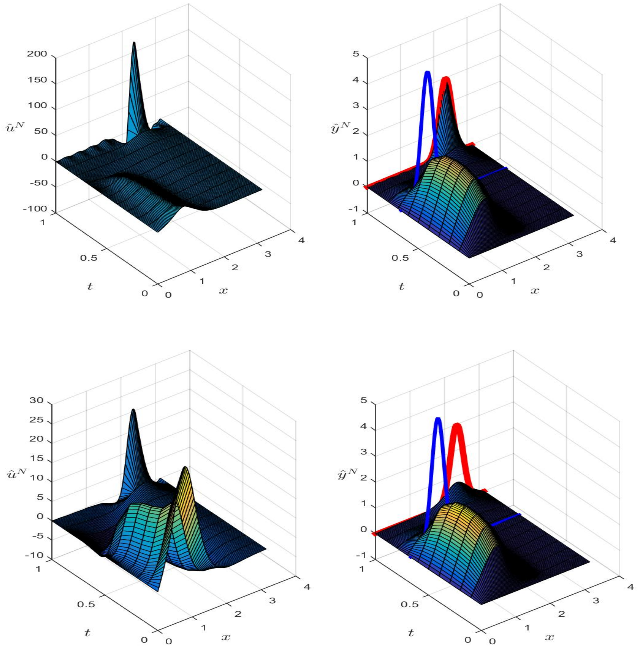

$\alpha = 0.01; \varepsilon^2 = [\ 0.5,\ 0.99999\ ] \cdot g(0).$

Conclusion

The new approach:

- exploring spectral representation of the solution by eigenfunctions of $\mathcal{A}$,
- an explicit expression of the optimal constrained control in terms of the given problem data.
- the numerical issues are reduced to finding the unique root of a scalar function.
- same formula applies indpendetnly of the dimension.
  No curse of dimensionality.

Price to pay:

- knowledge of eigenfunctions,
  If the problem has to be considered many times for different data, but the same operator, this can be done offline.
- BB^* diagonalisable
  This is not inevitable asumption, just the computations turn to be more delicate to handle without numerical tools.

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Thanks for your attention!