# Quantitative unique continuation principles for Schrödinger operators and its application to control theory


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# Aims of the talk

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# Aims of the talk

to give an introduction to unique continuation principles and their cousins: observability inequalities, uncertainty principles and spectral inequalities and their use in establishing controllability and estimating cost of controllability for control systems, especially for the control of heat equation, and

to present resent results about quantitative unique continuation principle for Schrödinger operators on unbounded domains and its application to control theory for the heat equation.

# Concepts of controllability

We consider the equation in the unknown $z : [0, T] \to H$

$$\dot{z} = Az + Bu,$$
$$z(0) = z_0.$$

Here $u$ is the control function and the solution $z$ of the equation is called state.
This equation models a control system.
Under mild conditions the solution is given by

$$z(t) = e^{tA} z_0 + \int_0^t e^{(t-s)A} Bu(s)\, ds, \text{ for all } 0 \leq t \leq T.$$

# Concepts of controllability

## Definition

The control system is exactly controllable in time $T$ if for any $z_0$, $z_T$ there exists $u$ such that the state $z$ satisfies $z(T) = z_T$.

The control system is null controllable in time $T$ if for any $z_0$ there exists $u$ such that the state satisfies $z(T) = 0$.

The control system is approximatively controllable in time $T$ if for any $z_0$, $z_T$ and any $\varepsilon > 0$ there exists $u$ such that the state $z$ satisfies $\|z(T) - z_T\| < \varepsilon$.

In the case of control systems modelled by PDEs, exactly controllability is too strong property.

# Checking controllability

We introduce the operator

$$L_T u = \int_0^T e^{(T-s)A} B u(s)\, ds.$$

Then $z(T) = e^{TA} z_0 + L_T u$ and we see that we have

$$\begin{aligned} \text{exact controllability in time } T &\iff R(L_T) = H \\ \text{null controllability in time } T &\iff R(e^{TA}) \subset R(L_T) \\ \text{approximate controllability in time } T &\iff \overline{R(L_T)} = H \end{aligned}$$

# The adjoint system

It is useful to introduce the adjoint system

$$\dot{\varphi} = -A^*\varphi,$$
$$\varphi(T) = \varphi_0.$$

This system is without control and backwards in time.
The solution is given by

$$\varphi(t) = e^{(T-t)A^*}\varphi_0.$$

The adjoint system is interesting for us because

$$L_T^* = B^* e^{(T-\cdot)A^*}$$

and we know that for operators $F, G$ we have

$$R(F) \subset R(G) \iff \|F^*\| \leq c\|G^*\|, c > 0.$$
$$\overline{R(F)} \subset \overline{R(G)} \iff N(F^*) \supset N(G^*).$$

# Controllability tests

## Theorem

*The system is null controllable in time T if and only if there exists a constant c* $> 0$ *such that*

$$\int_0^T \|B^* e^{tA^*} \varphi_0\|^2 \, dt \geq c \|e^{TA^*} \varphi_0\|, \text{ for all } \varphi_0.$$

This is called an observability inequality.

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### Theorem

*The system is approximately controllable in time T if and only if*

$$B^* e^{tA^*}\varphi_0 = 0 \text{ for all } 0 \leq t \leq T \iff \varphi_0 = 0.$$

This is called a unique continuation property (UCP).

# The heat equation

$$\begin{cases} \partial_t z - \Delta z = \mathbf{1}_\omega f & \text{in } \Omega \times [0,T] \\ z = 0 & \text{on } \partial\Omega \times [0,T] \\ z(0) = z_0 & \text{in } \Omega \end{cases}$$

Its adjoint system is

$$\begin{cases} -\partial_t \varphi - \Delta \varphi = 0 & \text{in } \Omega \times [0,T] \\ \varphi = 0 & \text{on } \partial\Omega \times [0,T] \\ \varphi(T) = \varphi_0 & \text{in } \Omega \end{cases}$$

Here $A = A^* = \Delta$ and $B = \mathbf{1}_\omega$ with $\omega \subset \Omega$. It follows $B^*\varphi = \varphi_{|\omega}$.
The UCP now reads: if $\varphi$ is the solution of the adjoint equation and $\varphi = 0$ on $\omega \times [0,T]$ then $\varphi_0 = 0$.

# Unique continuation principle

The unique continuation property/principle for a differential equation states that if $z$ is the solution and $z = 0$ on $\Omega' \subset \Omega$, $\Omega'$ nice, then $z = 0$ on $\Omega$.

Hence, if our $\omega \times [0, T]$ is nice and the adjoint equation has a unique continuation property, then $\varphi = 0$ on $\omega \times [0, T]$ implies $\varphi = 0$ on $\Omega \times [0, T]$ which implies $\varphi_0 = 0$.

The heat equation has UCP for $\Omega$ open, bounded and connected and $\Omega'$ with positive measure, so it follows that the heat equation is approximately controllable for such $\Omega$, $\Omega'$.

# Quantitative UCP & control cost

Roughly: if solution $z$ is small on $\Omega' \subset \Omega$, $\Omega'$ nice, then $z$ should be not too large on $\Omega$. Or more precisely

$$\|z\|_{\Omega'} \geq C\|z\|_{\Omega}, \text{ where } C \text{ does not depend on } z.$$

Using quantitative UCP one can obtain more detailed information about the controllability.
Control cost for null controllability is defined as

$$\mathcal{C} = \mathcal{C}(T, z_0) = \inf\{\|f\| : z(T) = 0\}.$$

Control cost coincides with the infimum of all $\sqrt{C}$, $C$ from the observability inequality.
Using quantitative UCP one can obtain an estimate for the control cost $\mathcal{C}$.
Hence, there is an important connection between observability inequalities and quantitative UCPs .

# Spectral inequalities

The link between observability inequalities and quantitative UCPs is through a particular kind of quantitative UCP which are sometimes called spectral inequalities.

With $\chi_{(-\infty,E)}(A)$ we denote the spectral subspace of a selfadjoint operator $A$ corresponding to the part of the spectrum contained in $(-\infty, E)$. A spectral inequality has the form

$$\|\phi\|_{\Omega'} \geq C\|\phi\|_{\Omega}, \text{ for all } \phi \in R(\chi_{(-\infty,E)}(A)).$$

A spectral inequality gives us a quantitative UCP for the corresponding stationary equation, but instead of solution we are plugging in (roughly) linear combinations of eigenfunctions.

# Uncertainty principle

Another way of thinking about a spectral inequality

$$\|\phi\|_{\Omega'} \geq C\|\phi\|_{\Omega}, \text{ for all } \phi \in R(\chi_{(-\infty,E)}(A)).$$

is as a uncertainty relation:
condition $\phi \in R(\chi_{(-\infty,E)}(A))$ is a condition in momentum/Fourier–space, which then enforces delocalization/flatness in direct space.

# Scale--free quantitative UCP

For a real-valued $V \in L^\infty(\mathbb{R}^d)$ we define the self-adjoint operator $H$ on $L^2(\mathbb{R}^d)$ as

$$H := -\Delta + V.$$

## Theorem (NTTV)

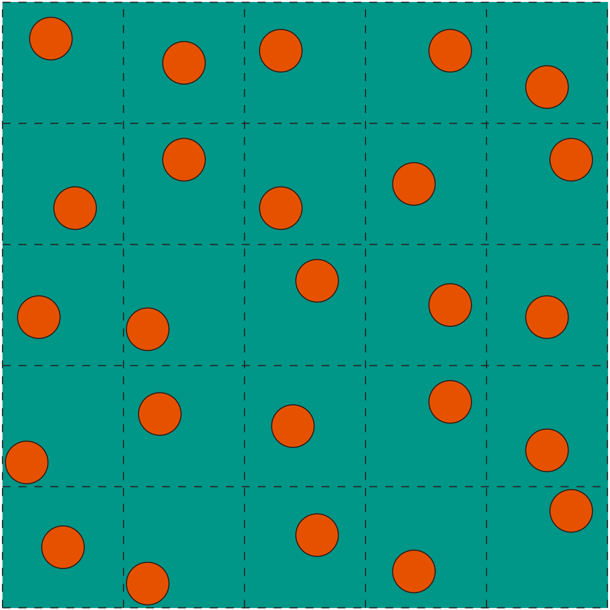
*For all $\delta \in (0, 1/2)$, all $(1, \delta)$-equidistributed sequences $Z$, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $E \geq 0$ and all $\psi \in R(\chi_{(-\infty,E)}(H))$ we have*

$$\|\psi\|^2_{L^2(S_{\delta,Z})} \geq C_{\mathrm{sfuc}} \|\psi\|^2_{L^2(\mathbb{R})}$$

*where*

$$C_{\mathrm{sfuc}} = C_{\mathrm{sfuc}}(d, \delta, E, \|V\|_\infty) := \delta^{N\left(1+\|V\|_\infty^{2/3}+\sqrt{E}\right)}.$$

# Geometric setting



# Remarks

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The domain does not have to be $\mathbb{R}^d$.

If the domain is bounded, the constant $C_{\text{sfuc}}$ does not depend on the size of the domain.

The constant $C_{\text{sfuc}}$ is explicit.

Instead of $\Delta$ we can have a 2nd order elliptic differential operator.

There is also a scaled version of the theorem, where the sizes of the little boxes are not $1$ but some $G > 0$.

For an unbounded domain, the spectral subspace $R(\chi_{(-\infty,E)}(H))$ is infinite–dimensional.

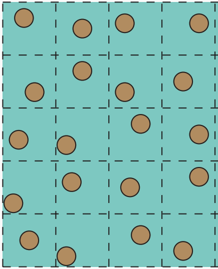
# (Other) applications

Used to prove so called Wegner estimates for random breather model, an ingredient towards the proof of Anderson localization in theory of random Schrödinger operators (models behaviour of e.g. alloys).

Used to prove eigenvalue movement and the movement of spectral edges in the gaps of the spectrum of a Schrödinger operator.

# Application in control theory

$$\begin{cases} \partial_t z - \Delta z + Vz = \mathbb{1}_\omega f & \text{in } \mathbb{R}^d \times [0,T] \\ z(0) = z_0 & \text{in } \mathbb{R}^d \end{cases}$$



$-V$ heat generation, $\omega = S_{\delta,Z}$

## Theorem (NTTV)

*For any $\delta \in (0, 1/2)$, any $V \in L^\infty(\mathbb{R}^d)$, any $(1,\delta)$-equidistributed sequence, and any $z_0 \in L^2(\mathbb{R}^d)$ the system is null-controllable with cost*

$$\mathcal{C} \leq C_1 e^{C_2/T} \|z_0\|$$

*with explicit $C_1$, $C_2$.*

# Remarks

The fact that such systems with unbounded domains are controllable is not trivial. The characterization of control sets for which the system with unbounded domain is controllable is still not known (except in the case $V = 0$ with the full space domain).

All the earlier remarks still valid. Hence, in the bounded case, the control cost does not depend on the size of the domain. Hence one can concatenating domains and still have the same control cost.

Another application is to have a fixed domain and scale the control subsets. For example, keeping the fraction $\delta/G$ constant.

# Techniques used in the proof

Quantitative UCP is proved using two interpolation inequalities of the form

$$\|\psi\|_{X_2} \leq C\|\psi\|_{X_1}^{\alpha}\|\psi\|_{X_3}^{1-\alpha}, \; X_1 \subset X_2 \subset X_3,$$

obtained from two different Carleman inequalities with explicit dependence on parameters, together with a careful covering of the domain and a chaining argument. The interpolation inequalities are applied to the function $\Psi$, constructed from $\psi \in R(\chi_{(-\infty,E)}(H))$, such that $\partial_t \Psi(0) = \psi$, and such that it is an eigenfunction of a new PDE in a higher dimension.

The control result is proved using a generalization of Tenenbaum & Tucsnak 2011 to a non–discrete setting.

# Thanks for the attention!