Approximation bounds for parameter dependent quadratic eigenvalue problem

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ConDyS

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Consider a damped linear vibrating system (stationary case)

$$M\ddot{x} + D\dot{x} + Kx = 0,$$
$$x(0) = x_0, \quad and \quad \dot{x}(0) = \dot{x}_0,$$

where $M, D, K \in \mathbb{R}^{n\times n}$ (mass, damping, stiffness),
M and $K > 0$ positive definite.

$D = C_{int} + C_{ext}$, where $C_{ext} \geq 0$ is external (viscous) damping.

C_{int} internal damping e.g. $C_{int} = \alpha_c C_{crit}$, where

$$C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}$$

Quadratic eigenvalue problem:

$$(\lambda^2 M + \lambda D + K)x = 0.$$

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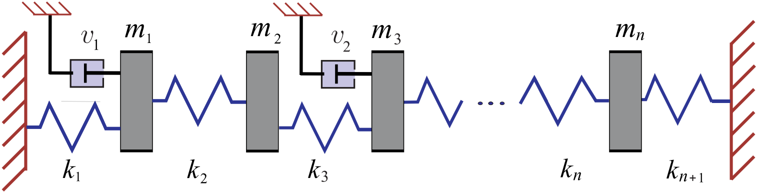
Example: *n-mass oscillator or oscillator ladder*



$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n),\ D = C_{int} + v_1 e_1 e_1^T + v_2 e_3 e_3^T.$$

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix},$$

$m_i > 0$ -ith mass, $k_i > 0$ -ith stiffness, v_1, v_2 - viscosities.

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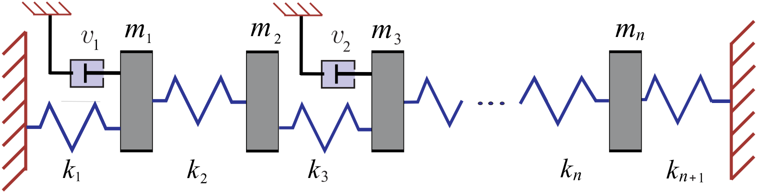

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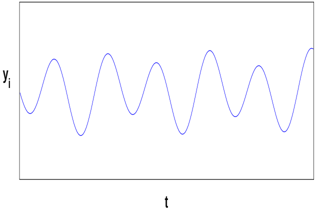
Very important question arises in considering such systems:

For the given mass (M) and stiffness (K) determine the best (optimal) damping which will insure optimal evanescence.

(M, K)



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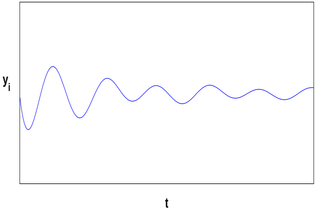

Spectral abscissa criterion:

$$\max_k \operatorname{Re}(\lambda_k) \quad \rightarrow \quad \min,$$

where λ_k are the complex eigenvalues of $\left(\lambda^2 M + \lambda D + K\right) x = 0.$

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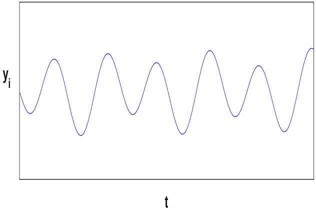

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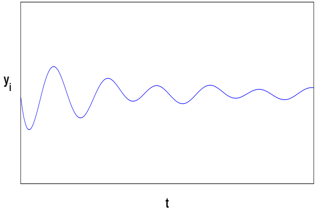

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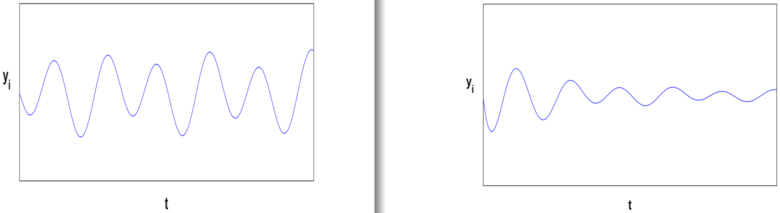

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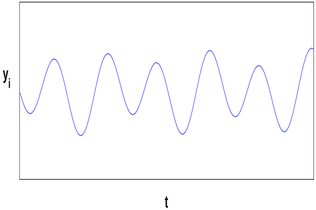

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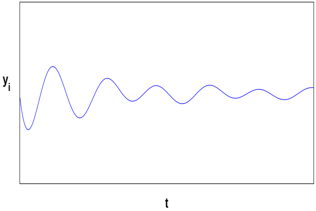

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Direct Velocity Feedback (DVF):

$$M\ddot{x} + Kx = bu, \qquad \text{structure equation}$$
$$y = b^T\dot{x}, \qquad \text{output equation (velocity sensor)}$$
$$u = -vy. \qquad \text{control equation}$$

Find $v \in \mathbb{R}$, s.t. *the resonance peaks are minimized.*

Mechanical systems with external force (non-stationary case):

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t)$$

Frequency isolation problem: re-design of a given damped mechanical system, such that a new system does not have eigenvalues in some "dangerous" part of a complex plane, typically called the resonance band.

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The first problem: Tracking eigenvalues

describe the behavior of the all eigenvalues or just of the part of spectrum for a different varieties of $\mathbf{v}$

The second poblem: For considered QEP

$$(\lambda^2 M + \lambda D + K)x = 0\,,$$

where $M, D, K \in \mathbb{C}^{n\times n}$ are Hermitian matrices.
and corresponding perturbed QEP

$$(\widetilde{\lambda}^2(M+\delta M) + \widetilde{\lambda}(D+\delta D) + K + \delta K)\widetilde{x} = 0,$$

where $M+\delta M$, $D+\delta D$ and $K+\delta K$ are also Hermitian.

i) bounds for difference of “appropriate scalar products”

ii) $sin\Theta$ bound

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Eigenvalue behavior $\lambda(v)$:

For $0 \leq \mathbf{v} \leq v_i \ll 1, i = 1, \ldots, k$: eigenvalues $\lambda(\mathbf{v})$ can be efficiently approximated by *modal approximation* [1].
For $\mathbf{v} \geq v_i \gg 1, i = 1, \ldots, k$: eigenvalues $\lambda(\mathbf{v})$ can be efficiently approximated using [2].

The new result:

We present an efficient way for tracking $\lambda(\mathbf{v}), \mathbf{v} = [v_1, \ldots, v_k]$, for $0 \leq v_i \leq V_M, i = 1, \ldots, k$.
Here V_M is of modest (even arbitrary) magnitude.

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We consider a damped linear vibrating system

$$M\ddot{x} + D\dot{x} + Kx = 0$$

with $M, D, K > 0$.

Linearization

- Let Φ simultaneously diagonalizes pair (M, K)

$$\Phi^T K \Phi = \Omega^2 = \operatorname{diag}(\omega_1^2, \dots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I.$$

With $x = \Phi x_\Phi$ and $y_1 = \Omega x_\Phi$, $y_2 = \dot{x}_\Phi$ we have

$$\frac{d}{dt}\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D(\mathbf{v})\Phi \end{bmatrix}}_{A(\mathbf{v})} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

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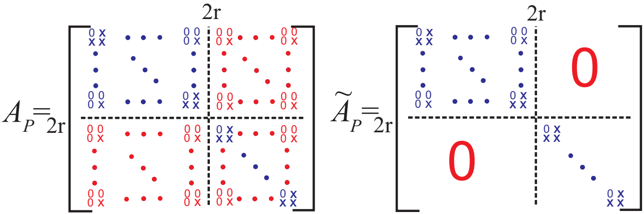

If $\|C(:, r+1:n)\| (= \|C(r+1:n, :)\|)$ is small.

We approximate A_P (after the perfect shuffle permutation) with $\widetilde{A}_P$

Hopping that: $r \ll n$, approximate $A(\mathbf{v})y(\mathbf{v}) = \lambda(\mathbf{v})y(\mathbf{v})$ with $\widetilde{A}_P(\mathbf{v})\widetilde{y}(\mathbf{v}) = \widetilde{\lambda}(\mathbf{v})\widetilde{y}(\mathbf{v})$.

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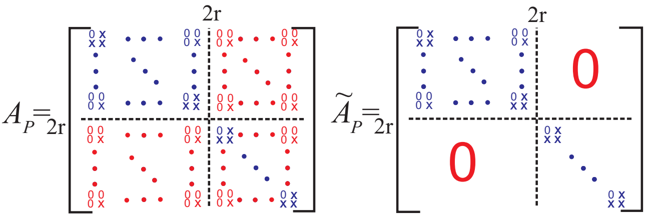

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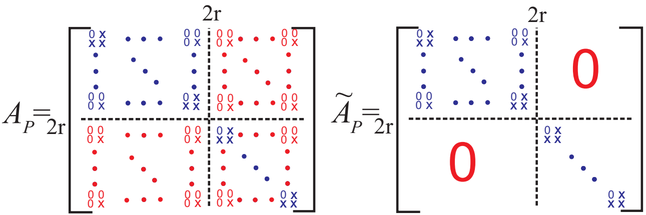

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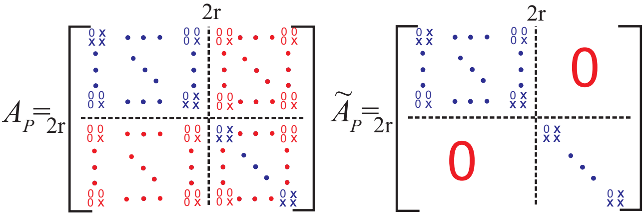

If $\|C(:, r+1:n)\| (= \|C(r+1:n, :)\|)$ is small.

We approximate A_P (after the perfect shuffle permutation) with $\widetilde{A}_P$

Hopping that: $r \ll n$, approximate $A(\mathbf{v})y(\mathbf{v}) = \lambda(\mathbf{v})y(\mathbf{v})$
with $\widetilde{A}_P(\mathbf{v})\widetilde{y}(\mathbf{v}) = \widetilde{\lambda}(\mathbf{v})\widetilde{y}(\mathbf{v})$.

Questions:

How to choose reduced dimension r?

What is the error made by this approximation?

How reduced dimension r and the error depend on parameters v?

Answers:

For the reduced dimension r we use approximation derived for damping of all eigenfrequencies [3] and for damping only selected eigenfrequencies [4].

The error and eigenvalue approximation will be presented.

[3] P. Benner, Z. Tomljanović, N. Truhar, Dimension reduction for damping optimization in linear vibrating system, Journal of Applied Mathematics and Mechanics 91/3 (2011), 179-191

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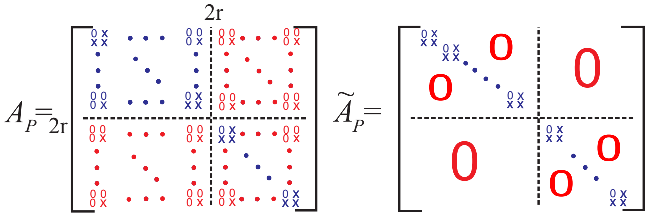
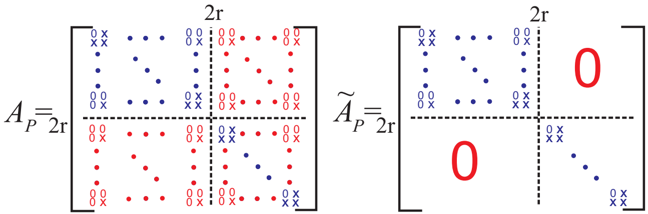


The new approach for approximation of all eigenvalues



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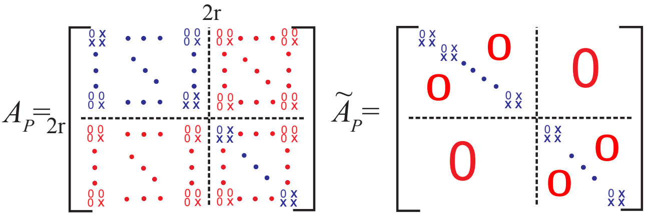

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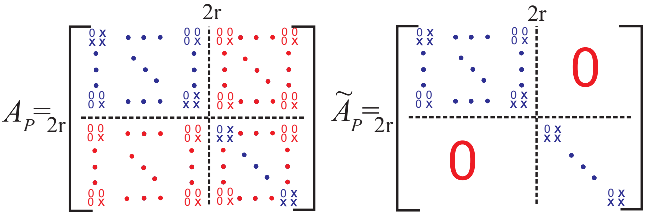

The error bounds (tracking all eigenvalues)

We approximate matrix $\widetilde{A}_P$ by the matrix

$$\overline{A}_P = \begin{bmatrix} \overline{A}_{11} & 0 \\ 0 & \overline{A}_{22} \end{bmatrix},$$

with

$$\overline{A}_{11} = \widetilde{A}_P(1:2r, 1:2r),$$

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The matrix Ψ^J_i has eigenvalues

$$\widetilde{\lambda} = \frac{-\gamma_i - C_{ii} \pm \sqrt{(\gamma_i + C_{ii})^2 - 4\omega_i^2}}{2} \quad \text{for } i = r+1, \ldots, n,$$

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Moreover, we need to diagonalize all two-by-two matrices $\Psi^J_{w(i)}$.

We assume that $(\gamma_w(i) + C_{w(i)w(i)})^2 - 4\omega^2_{w(i)} \neq 0$; thus matrices $Y_{r+1,r+1}, \ldots, Y_{n,n}$ diagonalize $\Psi^J_{w(i)}$.

$$\Psi^J_{w(i)} = Y_{ii}\,\mathrm{diag}(\lambda_{2i-1}, \lambda_{2i})Y_{ii}^{-1}, \quad \forall i = r+1, \ldots, n.$$

Then, using a block diagonal matrix

$$\hat{X} = \begin{bmatrix} X_{11} & 0 \\ 0 & \bigoplus_{i=1}^{n-k} Y_{ii} \end{bmatrix}$$

we obtain

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Gerschgorin type bound gives:

$$|\widetilde{\lambda}_i - \lambda_{\pi(i)}(\hat{A}_P)| \leq \sum_{j=r+1}^{2n} \left|(X_{11}^{-1}\widetilde{A}_{12} \oplus_{l=r+1}^{n} Y_{ll})_{ij}\right|, \quad i = 1, \ldots, 2r,$$

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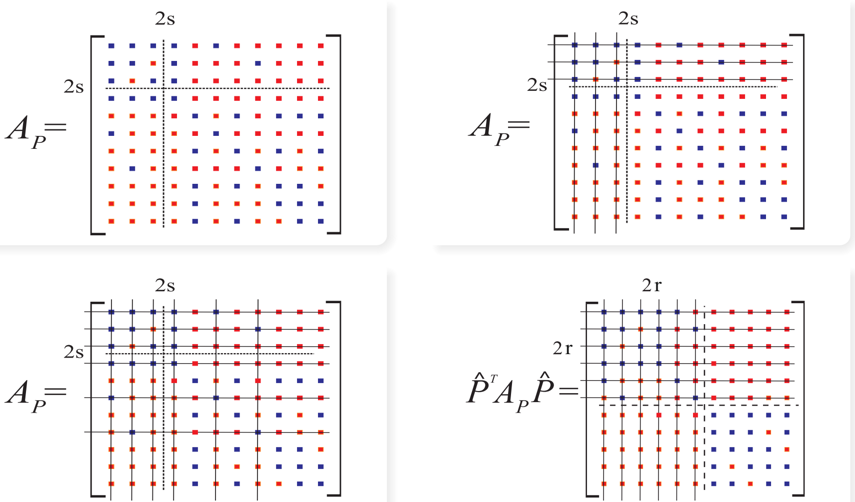
Approximation of the selected eigenvalues

Idea:



Approximation of the selected eigenvalues

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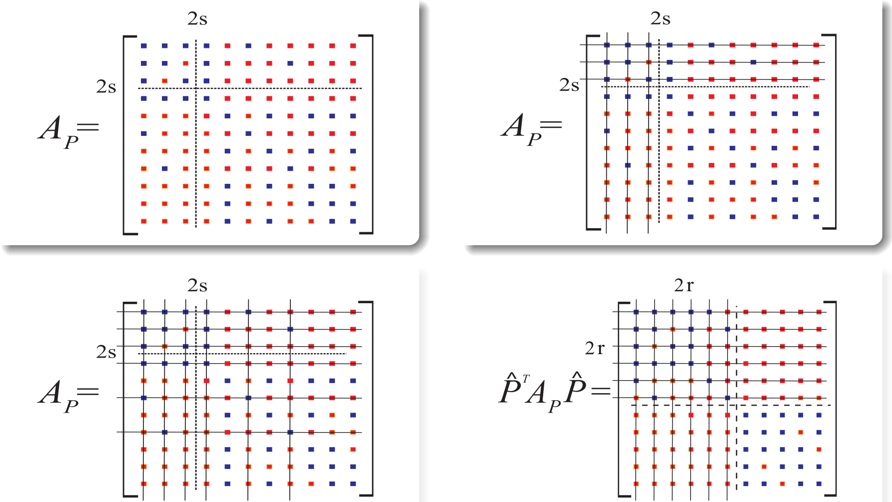

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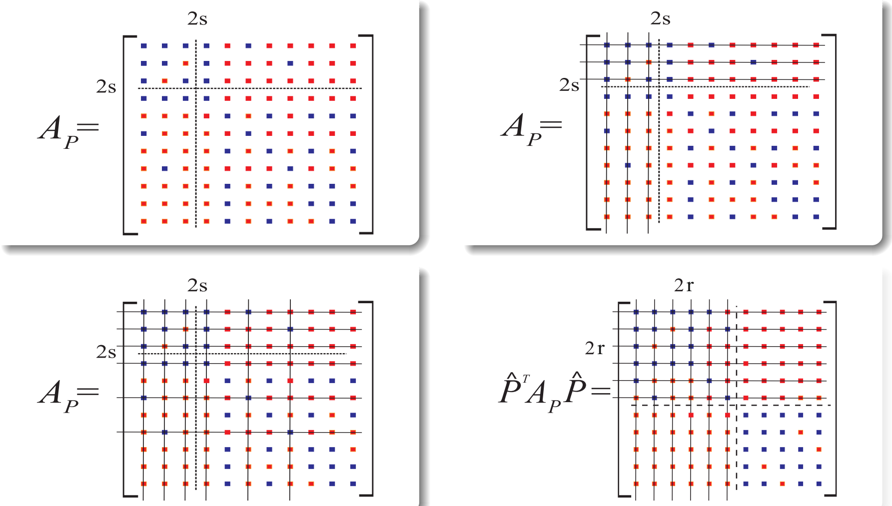

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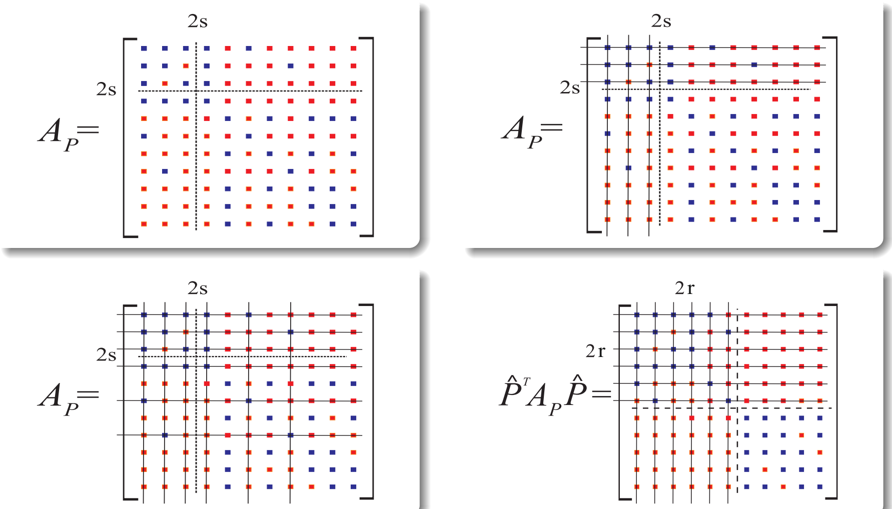

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If $\eta_2(\widetilde{\lambda}_i) = \min_{\mu \in \mathrm{eig}(\widetilde{A}_{22})} |\widetilde{\lambda}_i - \mu| > 0$ with matrices X_{11}, Y_{11} which diagonalise $\widetilde{A}_{11}$, $\widetilde{A}_{22}$, resp., we have that [5]

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[5] R. Li, Y. Nakatsukasa, N. Truhar, S. Xu, Perturbation of partitioned hermitian generalized eigenvalue problem, SIAM Journal on Matrix Analysis and Applications 32 (2) (2011) 642–663.

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Example I: Tracking all eigenvalues

We consider oscillator ladder with two dampers, with configuration

$$n = 1000; \quad k_i = 1, \quad \forall i; \quad m_i = \begin{cases} 1200 - 2i, & i = 1, \dots, 200, \\ 4i, & i = 201, \dots, n. \end{cases}$$

$$D = C_u + C_{ext}, \quad \text{with,} \quad C_{ext} = v e_{600} e_{600}^T + \frac{v}{4} e_{900} e_{900}^T.$$

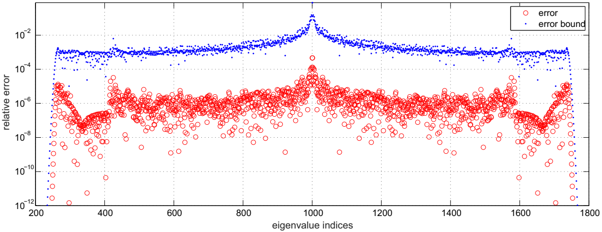
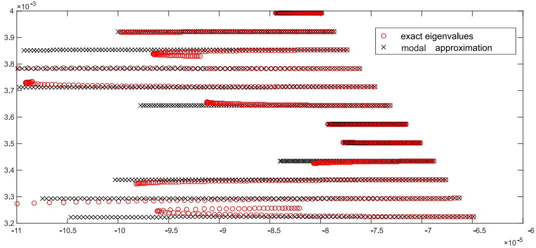
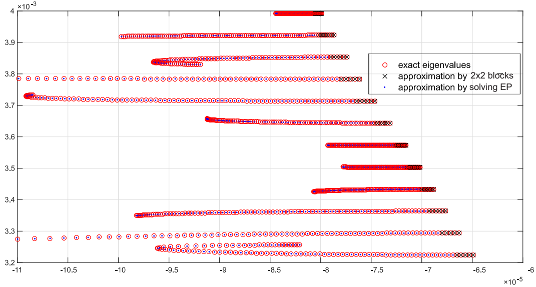


Figure: Relative error for $v = 10$ with $r = 416$.

Eigenvalue behaviour for $(v/4, v)$, $v = 1, \ldots, 100$.





Example II: Tracking selected eigenvalues

Consider $3d+1$-*mass oscillator*

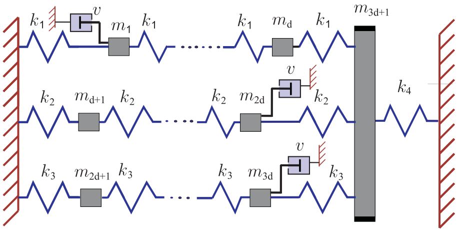


Figure: $3d+1$-mass oscillator with 3 dampers

$$d = 400, \quad n = 3d + 1 = 1201,$$

$$m_k = k, \quad k = 1, \ldots, n,$$

$$k_1 = 1, \quad k_2 = 20, \quad k_3 = 40, \quad k_4 = 50.$$

$$D = C_u + C_{ext}, \quad \text{with,} \quad C_{ext} = ve_{350}e_{350}^T + ve_{600}e_{600}^T + ve_{1000}e_{1000}^T.$$

Example II: Tracking selected eigenvalues

Consider $3d+1$-*mass oscillator*

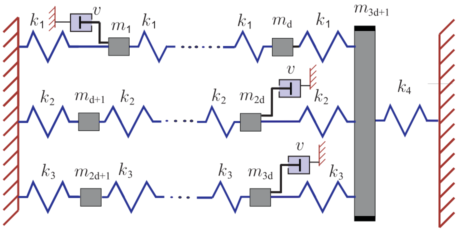


Figure: $3d+1$-mass oscillator with 3 dampers

$$d = 400, \quad n = 3d + 1 = 1201,$$

$$m_k = k, \quad k = 1, \ldots, n,$$

$$k_1 = 1, \quad k_2 = 20, \quad k_3 = 40, \quad k_4 = 50.$$

$$D = C_u + C_{ext}, \quad \text{with,} \quad C_{ext} = ve_{350}e_{350}^T + ve_{600}e_{600}^T + ve_{1000}e_{1000}^T.$$

$3d+1$-mass oscillator

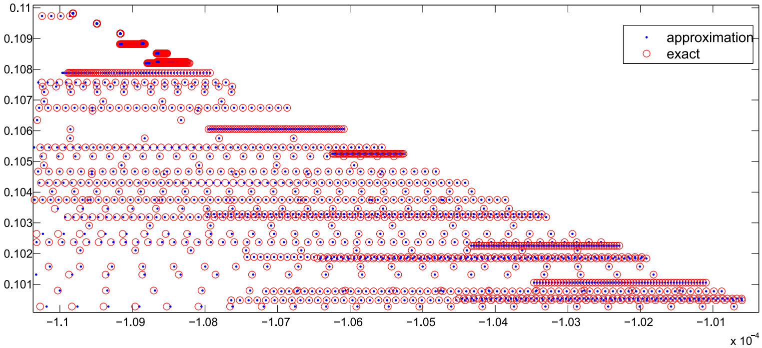
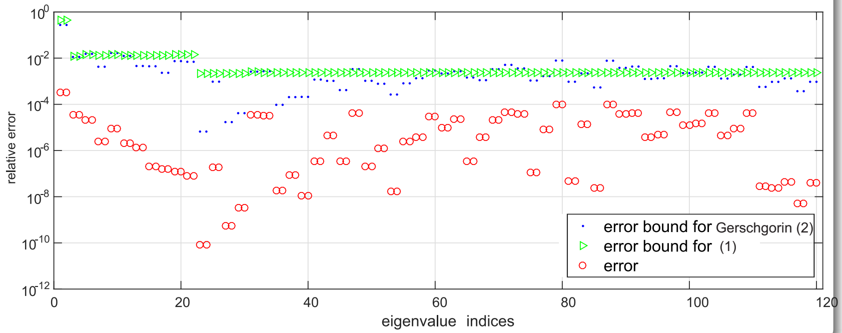


Figure: Eigenvalue behaviour for $0.05 \leq v \leq 3$, $0.1 \leq Im(\lambda(v)) \leq 0.11$

$3d + 1$-mass oscillator

Relative error for $v = 2, r = 60$.



Bounds on subspaces

Let $x_1, \dots, x_n$ be n linearly independent right (left) eigenvectors, and let $\lambda_1, \dots, \lambda_n$ be corresponding eigenvalues.
Interested in behaviour of $x_1, \dots, x_k$, which belong to $\lambda_1, \dots, \lambda_k$,

$$X = \left[X_1, X_2\right], X_1 = \left[x_1, \dots, x_k\right], X_2 = \left[x_{k+1}, \dots, x_n\right],$$
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corresponding perturbed quantities will be denoted by $\sim$.
Plan: derive an upper bound for the norm difference:

$$\left| \|X_2^* M \widetilde{X}_1\|_F^2 - \|X_2^* M X_1\|_F^2 \right|.$$

Measure for the difference in M-scalar product of two non M-orthogonal bases.

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We consider QEP and corresponding perturbed QEP

$$(\lambda^2 M + \lambda D + K)x = 0$$

$$\left(\widetilde{\lambda}^2(M + \delta M) + \widetilde{\lambda}(D + \delta D) + (K + \delta K)\right)\widetilde{x} = 0$$

Now, the following equalities hold :

$$\overline{\Lambda}^2 X^* M + \overline{\Lambda} X^* D + X^* K = 0,$$

$$M X \Lambda^2 + D X \Lambda + K X = 0.$$

By multiplying with X and X^* from $(2,1)$-th blocks we obtain:

$$\overline{\Lambda}_2{}^2 X_2^* M X_1 + \overline{\Lambda}_2 X_2^* D X_1 + X_2^* K X_1 = 0,$$

$$X_2^* M X_1 \Lambda_1^2 + X_2^* D X_1 \Lambda_1 + X_2^* K X_1 = 0.$$

which implies

$$(X_2^* M X_1)_{ij} = -\frac{(X_2^* D X_1)_{ij}}{(\overline{\Lambda}_2)_{ii} + (\Lambda_1)_{jj}},$$

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which implies

$$(X_2^* M X_1)_{ij} = -\frac{(X_2^* D X_1)_{ij}}{(\overline{\Lambda}_2)_{ii} + (\Lambda_1)_{jj}},$$

Similarly from perturbed equation we obtain

$$\overline{\Lambda}_2{}^2 X_2^* M \widetilde{X}_1 + \overline{\Lambda}_2 X_2^* D \widetilde{X}_1 + X_2^* K \widetilde{X}_1 = 0\,, \tag{3}$$

$$X_2^* \widetilde{M} \widetilde{X}_1 \widetilde{\Lambda}_1^2 + X_2^* \widetilde{D} \widetilde{X}_1 \widetilde{\Lambda}_1 + X_2^* \widetilde{K} \widetilde{X}_1 = 0.$$

From the above equalities, there follows:

$$(X_2^* M \widetilde{X}_1)_{ij} = -\frac{\left(X_2^* D \widetilde{X}_1\right)_{ij}}{(\overline{\Lambda}_2)_{ii} + (\widetilde{\Lambda}_1)_{jj}} + \frac{\left(X_2^* \delta M \widetilde{X}_1 \widetilde{\Lambda}_1^2\right)_{ij}}{(\overline{\Lambda}_2)_{ii}^2 + (\widetilde{\Lambda}_1)_{jj}^2} + \frac{\left(X_2^* \delta D \widetilde{X}_1 \widetilde{\Lambda}_1\right)_{ij}}{(\overline{\Lambda}_2)_{ii}^2 - (\widetilde{\Lambda}_1)_{jj}^2} + \frac{\left(X_2^* \delta K \widetilde{X}_1\right)_{ij}}{(\overline{\Lambda}_2)_{ii}^2 - (\widetilde{\Lambda}_1)_{jj}^2}. \tag{4}$$

Similarly from perturbed equation we obtain

$$\overline{\Lambda}_2{}^2 X_2^* M \widetilde{X}_1 + \overline{\Lambda}_2 X_2^* D \widetilde{X}_1 + X_2^* K \widetilde{X}_1 = 0\,, \tag{3}$$

$$X_2^* \widetilde{M} \widetilde{X}_1 \widetilde{\Lambda}_1^2 + X_2^* \widetilde{D} \widetilde{X}_1 \widetilde{\Lambda}_1 + X_2^* \widetilde{K} \widetilde{X}_1 = 0.$$

From the above equalities, there follows:

$$\begin{aligned}(X_2^* M \widetilde{X}_1)_{ij} = &- \frac{\left(X_2^* D \widetilde{X}_1\right)_{ij}}{(\overline{\Lambda}_2)_{ii} + (\widetilde{\Lambda}_1)_{jj}} + \frac{\left(X_2^* \delta M \widetilde{X}_1 \widetilde{\Lambda}_1^2\right)_{ij}}{(\overline{\Lambda}_2)_{ii}^2 + (\widetilde{\Lambda}_1)_{jj}^2} \\ &+ \frac{\left(X_2^* \delta D \widetilde{X}_1 \widetilde{\Lambda}_1\right)_{ij}}{(\overline{\Lambda}_2)_{ii}^2 - (\widetilde{\Lambda}_1)_{jj}^2} + \frac{\left(X_2^* \delta K \widetilde{X}_1\right)_{ij}}{(\overline{\Lambda}_2)_{ii}^2 - (\widetilde{\Lambda}_1)_{jj}^2}.\end{aligned} \tag{4}$$

$$
\begin{aligned}
\left| \|X_2^* M \widetilde{X}_1\|_F^2 - \|X_2^* M X_1\|_F^2 \right| \leq & \left| \frac{\|X_2^* D \widetilde{X}_1\|_F^2}{\widetilde{rg}_0^2} - \frac{\|X_2 D X_1\|_F^2|}{rg_0^2} \right| + \\
+2 \sum_{i,j} & \left| \frac{\left(X_2^* D \widetilde{X}_1\right)_{ij}}{\widetilde{rg}_0} \left(\frac{\left(X_2^* \delta M \widetilde{X}_1\right)_{ij}}{rg_1} + \frac{\left(X_2^* \delta D \widetilde{X}_1\right)_{ij}}{rg_2} + \frac{\left(X_2^* \delta K \widetilde{X}_1\right)_{ij}}{rg_3} \right) \right| \\
+ & \left| \left(\frac{\left(X_2^* \delta M \widetilde{X}_1\right)_{ij}}{rg_1} + \frac{\left(X_2^* \delta D \widetilde{X}_1\right)_{ij}}{rg_2} + \frac{\left(X_2^* \delta K \widetilde{X}_1\right)_{ij}}{rg_3} \right)^2 \right|.
\end{aligned}
$$

Where gaps are given by $\widetilde{rg}_0 = \min\limits_{\substack{i=1,\ldots,n-k \\ j=1,\ldots,k}} |(\Lambda_2)_{ii} + (\widetilde{\Lambda}_1)_{jj}|,$

$$
rg_0 = \min_{\substack{i=1,\ldots,n-k \\ j=1,\ldots,k}} |(\Lambda_2)_{ii} + (\Lambda_1)_{jj}|, \qquad rg_1 = \min_{\substack{i=1,\ldots,n-k \\ j=1,\ldots,k}} \frac{|(\Lambda_2)_{ii}^2 - (\widetilde{\Lambda_1})_{jj}^2|}{|(\widetilde{\Lambda_1})_{jj}^2|},
$$

$$
rg_2 = \min_{\substack{i=1,\ldots,n-k \\ j=1,\ldots,k}} \frac{|(\Lambda_2)_{ii}^2 - (\widetilde{\Lambda_1})_{jj}^2|}{|(\widetilde{\Lambda_1})_{jj}|}, \qquad rg_3 = \min_{\substack{i=1,\ldots,n-k \\ j=1,\ldots,k}} |(\Lambda_2)_{ii}^2 - (\widetilde{\Lambda_1})_{jj}^2|.
$$

$$
\begin{aligned}
&\left|\|X_2^* M \widetilde{X}_1\|_F^2 - \|X_2^* M X_1\|_F^2\right| \leq \left| \frac{\|X_2^* D \widetilde{X}_1\|_F^2}{\widetilde{rg}_0^2} - \frac{\|X_2 D X_1\|_F^2|}{rg_0^2} \right| + \\
&+2\sum_{i,j} \left| \frac{\left(X_2^* D \widetilde{X}_1\right)_{ij}}{\widetilde{rg}_0} \left(\frac{\left(X_2^* \delta M \widetilde{X}_1\right)_{ij}}{rg_1} + \frac{\left(X_2^* \delta D \widetilde{X}_1\right)_{ij}}{rg_2} + \frac{\left(X_2^* \delta K \widetilde{X}_1\right)_{ij}}{rg_3} \right) \right| \\
&+ \left| \left(\frac{\left(X_2^* \delta M \widetilde{X}_1\right)_{ij}}{rg_1} + \frac{\left(X_2^* \delta D \widetilde{X}_1\right)_{ij}}{rg_2} + \frac{\left(X_2^* \delta K \widetilde{X}_1\right)_{ij}}{rg_3} \right)^2 \right|.
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$$

$$
rg_2 = \min_{\substack{i=1,\ldots,n-k \\ j=1,\ldots,k}} \frac{|(\Lambda_2)_{ii}^2 - (\widetilde{\Lambda_1})_{jj}^2|}{|(\widetilde{\Lambda_1})_{jj}|}, \qquad rg_3 = \min_{\substack{i=1,\ldots,n-k \\ j=1,\ldots,k}} |(\Lambda_2)_{ii}^2 - (\widetilde{\Lambda_1})_{jj}^2|.
$$

Let M, D, K be simultaneously diagonalizable, this means that there exists a non singular matrix X such that

$$X^*MX = \Psi_M \doteq \operatorname{diag}(\psi_{(M,1)}, \psi_{(M,2)}, \ldots, \psi_{(M,n)}) \,,$$
$$X^*DX = \Psi_D \doteq \operatorname{diag}(\psi_{(D,1)}, \psi_{(D,2)}, \ldots, \psi_{(D,n)}) \,,$$
$$X^*KX = \Psi_K \doteq \operatorname{diag}(\psi_{(K,1)}, \psi_{(K,2)}, \ldots, \psi_{(K,n)}) \,.$$

Let denote $Y^* = X^{-1}$ and decompose

$$X = \left[X_1, X_2\right], \; X_1 = \left[x_1, \ldots, x_k\right], X_2 = \left[x_{k+1}, \ldots, x_n\right],$$
$$Y = \left[Y_1, Y_2\right], \; Y_1 = \left[y_1, \ldots, y_k\right], Y_2 = \left[y_{k+1}, \ldots, y_n\right],$$
$$\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2), \; \Lambda_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_k), \Lambda_2 = \operatorname{diag}(\lambda_{k+1}, \ldots, \lambda_n).$$

We consider $\mathcal{X}_1 = \operatorname{span}(M^{1/2}X_1)$ and $\widetilde{\mathcal{X}}_1 = \operatorname{span}(M^{1/2}\widetilde{X}_1)$.

The columns of X and Y span the same subspaces, thus if we denote

$$[X_1, X_2] = [Q_1, Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},$$

Since, $Y_2^* X_1 = 0$, using

$$Y_2 = \hat{Q}_2 \hat{R}_2, \quad \hat{Q}_2^* Q_1 = 0,$$

it can be shown that [Stewart, Sun, Matrix Perturbation Theory, 90]

$$\sigma_{\min}(\hat{R}_2)\sigma_{\min}(R_{11})\|\hat{Q}_2^* \widetilde{Q}_1\|_F \leq \|Y_2^* \widetilde{X}_1\|_F.$$

Using that $\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1) = \|\hat{Q}_2^* \widetilde{Q}_1\|_F$ we have

$$\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1) \leq \frac{1}{\sigma_{\min}(\hat{R}_2)\sigma_{\min}(R_{11})}\|Y_2^* \widetilde{X}_1\|_F.$$

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Using that $\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1) = \|\hat{Q}_2^* \widetilde{Q}_1\|_F$ we have

$$\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1) \le \frac{1}{\sigma_{\min}(\hat{R}_2)\sigma_{\min}(R_{11})} \|Y_2^* \widetilde{X}_1\|_F.$$

The columns x_k, $k = 1, \ldots, n$ are eigenvectors of QEP one can write

$$MX\Lambda^2 + DX\Lambda + KX = 0$$
$$\widetilde{M}\widetilde{X}\widetilde{\Lambda}^2 + \widetilde{D}\widetilde{X}\widetilde{\Lambda} + \widetilde{K}\widetilde{X} = 0\,,$$

for some $\widetilde{X}$ and corresponding perturbed eigenvalues $\widetilde{\Lambda}$.

Multiplying the perturbed equation with X^* from left one gets:

$$\underbrace{X^*M}_{\Psi_M X^{-1}}\widetilde{X}\widetilde{\Lambda}^2 + \underbrace{X^*D}_{\Psi_D X^{-1}}\widetilde{X}\widetilde{\Lambda} + \underbrace{X^*K}_{\Psi_K X^{-1}}\widetilde{X} = -\left(X^*\delta M\widetilde{X}\widetilde{\Lambda}^2 + X^*\delta D\widetilde{X}\widetilde{\Lambda} + X^*\delta K\widetilde{X}\right).$$

Now, from the above equality for the (i, j) component holds:

$$(Y^*\widetilde{X})_{ij}(\psi_{(M,i)}\widetilde{\lambda}_j^2 + \psi_{(D,i)}\widetilde{\lambda}_j + \psi_{(K,i)}) = -\left(X^*_{(:,i)}\delta M\widetilde{X}_{(:,j)}\widetilde{\lambda}_j^2 + X^*_{(:,i)}\delta D\widetilde{X}_{(:,j)}\widetilde{\lambda}_j + X^*_{(:,i)}\delta K\widetilde{X}_{(:,j)}\right).$$

The columns x_k, $k = 1, \ldots, n$ are eigenvectors of QEP one can write

$$M X \Lambda^2 + D X \Lambda + K X = 0$$
$$\widetilde{M} \widetilde{X} \widetilde{\Lambda}^2 + \widetilde{D} \widetilde{X} \widetilde{\Lambda} + \widetilde{K} \widetilde{X} = 0\,,$$

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Putting this altogether we can obtain upper bound for $\sin\Theta$

$$\sin\Theta(\mathcal{X}_1,\widetilde{\mathcal{X}}_1)^2 \leq \frac{\frac{Err(\widetilde{X}_1,X_1)}{\sigma_{\min}(\hat{R}_2)^2\sigma_{\min}(R_{11})^2}}{\min\limits_{\substack{1\leq j\leq k\\ k+1\leq i\leq n}} |\psi_{(M,i)}\widetilde{\lambda}_j^2+\psi_{(D,i)}\widetilde{\lambda}_j+\psi_{(K,i)}|^2}$$

where $Err(\widetilde{X}_1,X_1)=$

$$\sum_{i=k+1}^{n}\sum_{j=1}^{k}\left|X^*_{(:,i)}\delta M\widetilde{X}_{(:,j)}\widetilde{\lambda}_j^2+X^*_{(:,i)}\delta D\widetilde{X}_{(:,j)}\widetilde{\lambda}_j+X^*_{(:,i)}\delta K\widetilde{X}_{(:,j)}\right|^2$$

In a case of $\delta M=\delta K=0$, implies

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We consider example with configuration

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n), \quad m_i = 2i$$

$$K = \begin{pmatrix} 4k & -k & -k & & \\ -k & 4k & -k & -k & \\ & \ddots & \ddots & \ddots & \\ & -k & -k & 4k & -k \\ & & -k & -k & 4k \end{pmatrix} + K_0, \quad \text{with} \quad k = 0.1$$

K_0 determines the above perturbation term in the matrix K, where $k_0 = 5 \cdot 10^{-8}$

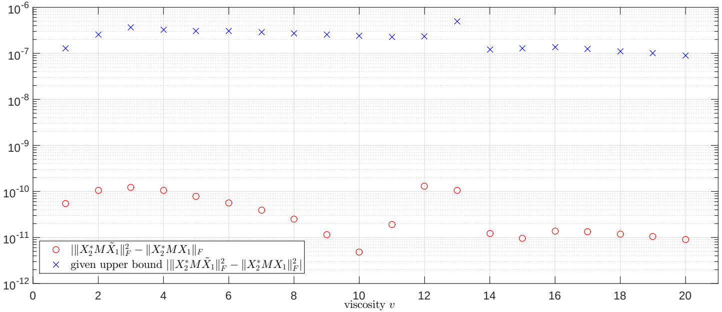
$$K_0(45:55, 45:55) = \begin{pmatrix} 2k_0 & 0 & -k_0 & & & \\ 0 & 2k_0 & 0 & -1k_0 & & \\ -1k_0 & 0 & 2k_0 & 0 & -1k_0 & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & -1k_0 & 0 & 2k_0 & 0 \\ & & & -1k_0 & 0 & 2k_0 \end{pmatrix}.$$

For Λ_2 we have taken 20 eigenvalues with the largest imaginary part, while for Λ_1 we have chosen 80 eigenvalues that come from 160 eigenvalues with the smallest, but positive imaginary part .

case I

Perturbation in the damping matrix is given by

$$\delta D = \delta v \, e_{60} e_{60}^T + \delta v \, e_{70} e_{70}^T \quad \text{with} \quad \delta v = 5 \cdot 10^{-6}.$$



case II

$$\delta D = \delta v \, e_j e_j^T \quad \text{with} \quad \delta v = 5 \cdot 10^{-4}.$$

for positions $j = 51, 52, \ldots, 70.$

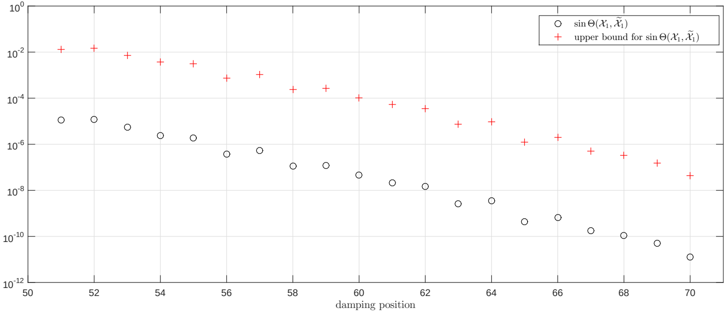


Figure: Numerical results for upper and lower bounds for $\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)$

Summary and outlook

1. within parameter dependent QEP:

The extension and application of modal approximation on

efficient approximation of eigenvalues $\lambda_i(v)$ for QEP $(\mu^2 M + \mu C(v) + K)x(v) = 0$.

♢ for all $i = 1, \ldots, n$ and modest $0 < v \leq V_M$.

♢ for the part of spectrum, i.e. $r_0 \leq i \leq r_0 + k$ and modest $0 < v \leq V_M$.

2. within perturbation theory:

♢ bound for the norm difference $\left| \|X_2^* M \widetilde{X}_1\|_F^2 - \|X_2^* M X_1\|_F^2 \right|$

♢ $\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)$ bound, $\mathcal{X}_1 = \operatorname{span}(M^{1/2} X_1)$, $\widetilde{\mathcal{X}}_1 = \operatorname{span}(M^{1/2} \widetilde{X}_1)$.

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Thank you for your attention!